Sequences

Def: $a_0, a_1, a_2, a_3, \ldots, a_n, \ldots$

a_0: first term

a_n: "general term"

Def: Geometric Sequence:

$\frac{a_{n+1}}{a_n} = r$ = constant ratio of consecutive terms (for all n)

Therefore: $a_0 \xrightarrow{\times r} a_1 \xrightarrow{\times r} a_2 \xrightarrow{\times r} a_3 \xrightarrow{\times r} a_4 \ldots$

$a_1 = a_0 r$

$a_2 = a_1 r = a_0 r^2$

$a_3 = a_2 r = a_1 r^2 = a_0 r^3$

i.e. $a_n = a_0 r^n$ "general term formula"

Examples: $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, 1 \cdot \left(\frac{1}{2}\right)^n, \ldots$

$a_0 = 1, \quad r = 1/2$

$\pi, \pi^2, \pi^3, \ldots, 1 \cdot \pi^n, \ldots ; \quad r = \pi$

$5, -\frac{1}{8}, \frac{1}{320}, \ldots$ $\quad r = \frac{-1/8}{5} = -\frac{1}{40}$

$\rightsquigarrow \left(-\frac{1}{8}\right)\left(-\frac{1}{40}\right) = \frac{1}{320}$

$\frac{3}{2}, -\frac{1}{2}, \frac{1}{6}, -\frac{1}{18}, \ldots$ $\quad r = \frac{-1/2}{3/2} = -\frac{1}{3}$

$\rightsquigarrow \left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right) = \frac{1}{6}$

$\rightsquigarrow \left(\frac{1}{6}\right)\left(-\frac{1}{3}\right) = -\frac{1}{18}$

$a_n = a_0 r^n$

Examples: (a) $a_n = \left(-\frac{1}{2}\right)^n$ (b) $b_n = -3\left(\frac{2^{n+1}}{5^n}\right)$

(c) $C_n = e^{5+7n}$. Find first and r. Write general term in standard form.

Solution: (a) $a_0 = 1$, $r = -\frac{1}{2}$, $a_n = 1 \cdot \left(-\frac{1}{2}\right)^n$

(b) $b_0 = -3\left(\frac{2^{0+1}}{5^0}\right) = -3\left(\frac{2}{1}\right) = -6$

$$\frac{b_{n+1}}{b_n} = \frac{-3\left(\frac{2^{n+1+1}}{5^{n+1}}\right)}{-3\left(\frac{2^{n+1}}{5^n}\right)} = \frac{2^{n+2}}{5^{n+1}} \cdot \frac{5^n}{2^{n+1}} = \frac{2}{5} = r$$

$$b_n = -6\left(\frac{2}{5}\right)^n$$

(c) $C_0 = e^5$, $r = e^7$

$$e^{5+7n} = e^5 e^{7n} = \left(e^5\right)\left(e^7\right)^n$$

$C_0 \quad r$

$$C_0 = e^{5+7\cdot 0} = e^5$$

$$\frac{C_{n+1}}{C_n} = \frac{e^{5+7(n+1)}}{e^{5+7n}} = \frac{e^{5+7n+7}}{e^{5+7n}} = e^7$$

$e^{5+7n} \cdot e^7$

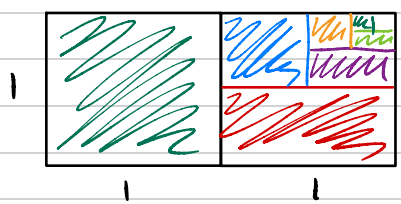
Series

Def: $\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \dots + a_n + \dots$

Examples: $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \dots = \frac{\pi}{4}$

"Leibniz Series" GOAT

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \left(\frac{1}{2}\right)^n + \dots = 2$$



Geometric Series
$a_0 = 1, \quad r = 1/2$

Total Sum of Geometric Series

Write "S" for the total sum:

$$S = a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \dots = \sum_{n=0}^{\infty} a_0 r^n$$

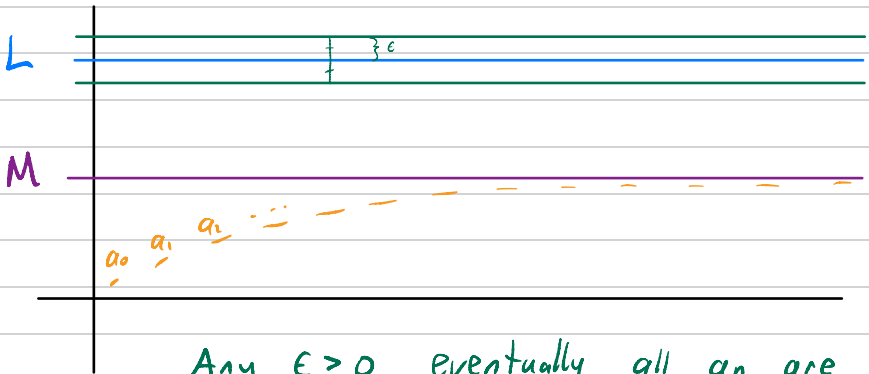
$$-\left(rS = a_0 r + a_0 r^2 + a_0 r^3 + a_0 r^4 + \dots\right)$$

$$S - rS = a_0 \quad \rightsquigarrow \quad S(1-r) = a_0$$

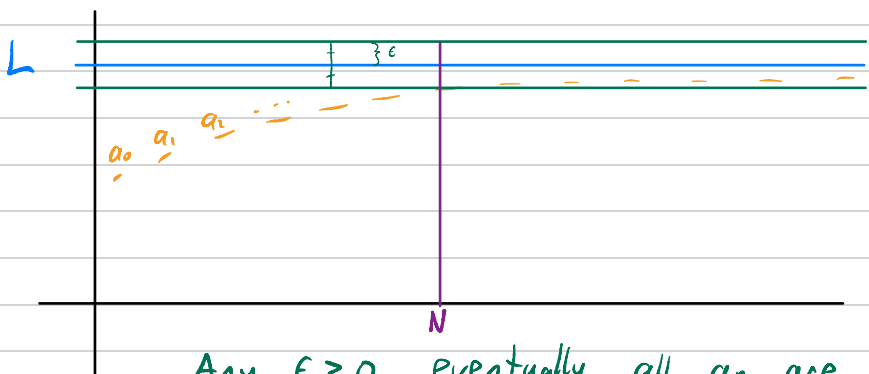
$$\rightsquigarrow \quad \boxed{S = \frac{a_0}{1-r}}$$

$S = \frac{1}{1-1/2}$
$= \frac{1}{1/2}$
$= \boxed{2}$

Convergence



Any $\epsilon > 0$, eventually all a_n are within ϵ of L.



Any $\epsilon > 0$, eventually all a_n are within ϵ of L.

i.e. some N such that $|a_n - L| < \epsilon$ when $n > N$.

write $\lim_{n \to \infty} a_n = L$

or write $a_n \to L$ as $n \to \infty$

L'Hopital's Rule

Replace n with x: $a_n \rightsquigarrow a_x$

Apply L'Hopital's to a_x. Same convergence valid for a_n.

Examples: (a) $\lim_{n\to\infty} \frac{\ln n}{n}$ (b) $\lim_{n\to\infty} \frac{(\ln n)^2}{n}$

(c) $\lim_{n\to\infty} n\left(\sqrt{n^2+1} - \sqrt{n^2}\right)$

Solutions: (a) $n \rightsquigarrow x$: $\frac{\ln x}{x} \to \frac{\infty}{\infty}$ indeterminate

L'H $\rightsquigarrow$ $\frac{1/x}{1} = \frac{1}{x} \to \boxed{0}$ as $x \to \infty$

(b) $\rightsquigarrow \frac{(\ln x)^2}{x} \overset{\text{L'H}}{\rightsquigarrow} \frac{2(\ln x)(1/x)}{1} = 2\frac{\ln x}{x} \to \boxed{0}$ by (a).

(c) $\sqrt{n^2+1} - \sqrt{n^2} \to 0$ (see monotonicity later)

$n\left(\sqrt{n^2+1} - \sqrt{n^2}\right) = \infty \cdot 0$ indeterminate.

$\hookrightarrow \frac{\sqrt{x^2+1} - \sqrt{x^2}}{1/x} \overset{\text{L'H}}{\rightsquigarrow} \frac{\frac{1}{2}(x^2+1)^{-1/2}(2x) - 1}{-1/x^2}$

$$\rightsquigarrow \quad \frac{-x^3}{\sqrt{x^2+1}} + \frac{1}{x^2} = \frac{-x^5 + \sqrt{x^2+1}}{x^2\sqrt{x^2+1}}$$

$x \to \infty$ $\rightsquigarrow$ $\frac{-x^5 + x}{x^2 \cdot x}$ $\rightsquigarrow$ $-\frac{x^5}{x^3} = -x^2 \to \boxed{-\infty}$

(d) $\lim_{n\to\infty} n e^{-n}$

$\infty \cdot 0$ indeterminate $\qquad x e^{-x} = \frac{x}{e^x} \overset{L'H}{\rightsquigarrow} \frac{1}{e^x} \to \boxed{0}$

Squeeze Theorem

Suppose $a_n \le b_n \le c_n$ all n
and $a_n \to L$, $c_n \to L$ as $n \to \infty$.

THEN $b_n \to L$ as $n \to \infty$ too.

Example: Show $\frac{4^n}{n!} \to 0$ as $n \to \infty$

Method: $0 \le \frac{4^n}{n!} \le c_n$ find c_n, $c_n \to 0$

Notice: $\frac{4^n}{n!} = \frac{4 \cdot 4 \cdot 4 \cdots 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{n \cdot (n-1)(n-2) \cdots 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$$\le \frac{4}{n} \cdot 1 \cdot 1 \cdot \ldots \cdot 1 \cdot 1 \cdot \frac{4}{4} \cdot \frac{4}{3} \cdot \frac{4}{2} \cdot \frac{4}{1}$$

$$= \frac{4}{n} \cdot \frac{4^4}{4!} = \frac{1024}{24n} \to 0 \text{ as } n \to \infty \quad \checkmark$$

c_n

Monotonicity Theorem

Monotone increasing: $a_{n+1} \geq a_n$ all n

monotone decreasing: $a_{n+1} \leq a_n$ all n

Theorem: If a_n is mon. incr. & bounded above, then a_n converges.

If a_n is mon. decr. & bounded below, then a_n converges.

⚠ Don't know L, just it exists.

Example: Show $a_n = \sqrt{n+1} - \sqrt{n}$ converges.

Proof: ○ Show decreases:

$n \rightsquigarrow x$: $\sqrt{x+1} - \sqrt{x} \xrightarrow{d/dx} \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}}$

$\rightsquigarrow \frac{2\sqrt{x} - 2\sqrt{x+1}}{4\sqrt{x(x+1)}} \rightsquigarrow -\frac{1}{2}\left(\frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x(x+1)}}\right)$ positive

$\rightsquigarrow$ negative $\Rightarrow$ decreasing.

○ $\sqrt{n+1} - \sqrt{n} > 0$ because $\sqrt{n+1} > \sqrt{n}$

$\Rightarrow$ bounded below by 0

$\Rightarrow$ by Monotonicity Theorem: a_n converges

Series Convergence

$\sum_{n=0}^{\infty} a_n$ converges $\iff$ S_N converges

$$S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \ldots + a_N$$

S_N = "partial sum sequence of the series"