W10 - Notes

Ratio test and Root test

Videos

Videos, Math Dr. Bob

- Ratio test: Basics
- Ratio test: Ratio test + DCT
- Root test: Basics
- Root test: for $\sum(1-1/n^2)^{n^3}$

01 Theory

Ratio Test (RaT)

Applicability: Any series with nonzero terms.

Test Statement:

Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$ as $n \to \infty$.

Then:

$$L < 1: \quad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely}$$

$$L > 1: \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

$$L = 1 \text{ or DNE}: \quad \text{test inconclusive}$$

Extra - Ratio Test - a deeper look

To understand the ratio test better, first consider this series:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots$$

- The term $\frac{2^3}{3!}$ is created by multiplying the prior term by $\frac{2}{3}$.
- The term $\frac{2^4}{4!}$ is created by multiplying the prior term by $\frac{2}{4}$.
- The term a_n is created by multiplying the prior term by $\frac{2}{n}$.

When $n > 3$, the multiplication factor giving the next term is necessarily less than $\frac{2}{3}$.
Therefore, when $n > 3$, the terms shrink *faster than those of a geometric series* having $r = \frac{2}{3}$.
Therefore this series converges.

Similarly, consider this series:

$$\sum_{n=0}^{\infty} \frac{10^n}{n!} \quad = \quad 1 + \frac{10}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots$$

Write $R_n = \frac{a_n}{a_{n-1}}$ for the ratio from the prior term a_{n-1} to the current term a_n. For this series, $R_n = \frac{10}{n}$.

This ratio falls below $\frac{10}{11}$ when $n > 11$, after which the terms necessarily shrink faster than those of a geometric series with $r = \frac{10}{11}$. Therefore this series converges.

The main point of the discussion can be stated like this:

$$R_n \to L < 1 \quad \text{as} \quad n \to \infty$$

Whenever this is the case, then *eventually* the ratios are bounded below some $r < 1$, and the series terms are smaller than those of a converging geometric series.

Extra - Ratio Test proof

Let us write $R_n = \left|\frac{a_{n+1}}{a_n}\right|$ for the ratio to the next term from term n.

Suppose that $R_n \to L$ as $n \to \infty$, and that $L < 1$. This means: eventually the ratio of terms is close to L; so eventually it is less than 1.

More specifically, let us define $r = \frac{L+1}{2}$. This is the point halfway between L and 1. Since $R_n \to L$, we know that eventually $R_n < r$.

Any geometric series with ratio r converges. Set $c = a_N$ for N big enough that $R_N < r$. Then the terms of our series satisfy $|a_{N+n}| \leq cr^n$, and the series starting from a_N is absolutely convergent by comparison to this geometric series.

(Note that the terms $a_1, \ldots, a_{N-1}$ do not affect convergence.)

02 Illustration

Example - Ratio test

(a) Observe that $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n+1}$ and thus $R_n \to 0 = L < 1$. Therefore the RaT implies that this series converges.

Simplify the ratio:

$$\frac{\left(\frac{10^{n+1}}{(n+1)!}\right)}{\left(\frac{10^n}{n!}\right)} \quad \gg\gg \quad \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$$

$$\gg\gg \quad \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \quad \gg\gg \quad \frac{10}{n+1} \overset{n\to\infty}{\longrightarrow} 0$$

Notice this technique! We *frequently* use these rules:

$$10^{n+1} = 10^n \cdot 10, \qquad (n+1)! = (n+1)n!$$

(To simplify ratios with exponents and factorials.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}$.

Simplify this:

$$\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \quad \gg\gg \quad \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\gg\gg \quad \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \quad \gg\gg \quad \frac{n^2+2n+1}{2n^2} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\displaystyle\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \frac{n^2+2n+1}{n^2} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ has ratio $R_n = \frac{n^2}{n^2+2n+1} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p-series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p-series.

03 Theory

Root Test (RooT)

Applicability: Any series.

Test Statement:

Suppose that $|a_n|^{1/n} \longrightarrow L$ as $n \to \infty$.

Then:

$$L < 1: \quad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely}$$

$$L > 1: \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

$$L = 1 \text{ or DNE}: \quad \text{test inconclusive}$$

Extra - Root test: explanation

The fact that $|a_n|^{1/n} \to L$ and $L < 1$ implies that eventually $|a_n|^{1/n} < \frac{L+1}{2}$ for all high enough n. Set $r = \frac{L+1}{2}$ (the midpoint between L and 1).

Now, the equation $|a_n|^{1/n} < r$ is equivalent to the equation $|a_n| < r^n$.

Therefore, eventually the terms $|a_n|$ are each less than the corresponding terms of this convergent geometric series:

$$\sum_{n=1}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$$

04 Illustration

Example - Root test examples

(a) Observe that $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0 = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

(b) Observe that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \overset{n\to\infty}{\longrightarrow} \frac{1}{2} = L$$

Because $L < 1$, the RooT shows that the series converges absolutely.

Example - Ratio test versus root test

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$a_{n+1} \cdot \frac{1}{a_n} = \left(\frac{n+1}{5}\right)^2 \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \left(\frac{5}{4}\right)^n$$

$$\gg\gg \qquad \frac{n^2+2n+1}{n^2} \cdot \frac{4}{5} \longrightarrow \frac{4}{5} < 1 \text{ as } n \to \infty$$

Therefore the series converges absolutely by the ratio test.

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \frac{4}{5}$$

To compute the limit as $n \to \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln\left(\left(\frac{n}{5}\right)^{2/n} \frac{4}{5}\right) = \frac{2}{n} \ln \frac{n}{5} + \ln \frac{4}{5}$$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln \frac{n}{5} \overset{d/dx}{\longrightarrow} \frac{1}{n/5}\frac{1}{5}}{n/2 \overset{d/dx}{\longrightarrow} 1/2} \qquad \gg\gg \qquad \frac{1/n}{1/2} \qquad \gg\gg \qquad \frac{2}{n} \longrightarrow 0 \text{ as } n \to \infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln \frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln \frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.

Series tests: strategy tips

Videos

Videos, Math Dr. Bob

- Series test round-up: Part I
- Series test round-up: Part II
- Series test round-up: Part III

Videos, Trefor Bazett

- How to choose a series convergence test

05 Theory

It can help to associate certain "strategy tips" to find convergence tests based on certain patterns.

Matching powers → Simple Divergence Test

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Use the SDT because we see the highest power is the *same* ($=1$) in numerator and denominator.

Rational or Algebraic → Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Use the LCT because we have a *rational or algebraic* function (positive terms).

Not rational, not factorials → Integral Test

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Use the IT because we do *not* have a rational/algebraic function, and we do *not* see factorials.

Rational, alternating → AST, and LCT or DCT

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$$

Notice large n behavior is like n^2/n^4 or $1/n^2$. This converges. Use the LCT to show absolute convergence. Skip the AST because absolute convergence settles the matter. Lesson: check for absolute BEFORE applying AST, even when alternating!

Factorials → Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Use the RaT because we see a factorial. (In case of alternating + factorial, use RaT first.)

Recognize geometric → LCT or DCT

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Use the LCT or DCT comparing to $\frac{1}{3^n}$ because we see similarity to $\frac{1}{3^n}$ (recognize geometric).

Power series: Radius and Interval

Videos

Videos, Math Dr. Bob

- Power series: Interval and Radius of Convergence
- Power series: Interval of Convergence Using Ratio Test
  - Further example
- Power series: Interval of Convergence Using Root Test
- Power series: Finding the Center

06 Theory

A power series looks like this:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Power series are used to *build and study functions*. They allow a uniform "modeling framework" in which many functions can be described and compared. Power series are also convenient for *computers* because they provide a way to store and evaluate *differentiable* functions with numerical (approximate) values.

⚠ Small x needed for power series

The most important fact about power series is that they work for *small values of x*.

Many power series diverge for $|x|$ too big; but even when they converge, for big $|x|$ they converge more slowly, and partial sum approximations are less accurate.

The idea of a power series is a modification of the idea of a geometric series in which the common ratio r becomes a variable x, and each term has an additional *coefficient parameter* a_n controlling the relative contribution of different orders.

07 Theory

Every power series has a **radius of convergence** and an **interval of convergence**.

Radius of convergence

Consider a power series centered at $x = 0$:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

Apply the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| \quad \gg\gg \quad \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|x| \;=\; L$$

Define the **radius of convergence** $R \in [0,\infty]$:

$$R \;=\; \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

Therefore:

$$|x| \;<R \quad\Longrightarrow\quad L<1 \quad\Longrightarrow\quad \text{converges}$$

$$|x| \;>R \quad\Longrightarrow\quad L>1 \quad\Longrightarrow\quad \text{diverges}$$

We can build **shifted power series** for x near some other value c. Just replace the variable x with a shifted variable $u = x - c$:

$$a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots$$

$$\gg\gg \quad a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Now apply the ratio test to determine convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}|x-c|^{n+1}}{a_n|x-c|^n}\right| \quad \gg\gg \quad \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)|x-c|$$

Define the **radius of convergence** $R \in [0,\infty]$:

$$R \;=\; \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

In the shifted setting, the radius of convergence limits the *distance from* :

$$|x-c| \;<R \quad\Longrightarrow\quad \text{converges}$$

$$|x-c| \;>R \quad\Longrightarrow\quad \text{diverges}$$

Method:
To calculate the **interval of convergence** of a power series, follow these steps:

- Observe the center c of the shifted series (or $c = 0$ for no shift).
- Compute R using the limit of coefficient ratios.
- Write down the *preliminary interval* $(c - R,\ c + R)$.
- Plug each endpoint, $c - R$ and $c - R$, into the original series
  - Check for convergence
- Add in the *convergent endpoints*. (4 possible scenarios.)

08 Illustration

Example - Radius of convergence

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$ (b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

(a) Ratio of terms:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}}\right|$$

$$\gg\gg \quad \left|\frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n}\right| \quad \gg\gg \quad \frac{1}{2}|x| = L$$

This converges for $L < 1$, or $|x| < 2$. Therefore $R = 2$.

(b) This power series skips the odd powers. Apply the ratio test to just the even powers:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right|$$

$$\gg\gg \quad \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right|$$

$$\gg\gg \quad \frac{1}{(2n+2)(2n+1)}|x^2| \quad \gg\gg \quad R = \infty$$

Example - Interval of convergence

Find the radius and interval of convergence of the following series:

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ (b) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(1) Apply ratio test:

$$\left|a_{n+1} \cdot a_n^{-1}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$\gg\gg \quad \frac{n}{n+1}|x-3| \quad \overset{n\to\infty}{\longrightarrow} \quad |x-3| = L$$

Therefore $R = 1$ and thus:

$$|x-3| < 1 \quad \Longrightarrow \quad \text{converges}$$

$$|x-3| > 1 \quad \Longrightarrow \quad \text{diverges}$$

Preliminary interval: $x \in (2, 4)$.

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(2) Check endpoints:

Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\gg\gg \quad \text{converges by AST}$$

Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\gg\gg \quad \text{diverges as } p\text{-series}$$

Final interval of convergence: $x \in [2, 4)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

(1) Apply ratio test:

$$\left|a_{n+1} \cdot a_n^{-1}\right| \; = \; \left| \frac{(-3)^{n+1} x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n} \right|$$

$$\gg\gg \quad \frac{|(-3)(-3)^n|}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{|(-3)^n|} |x|$$

$$\gg\gg \quad \frac{3\sqrt{n+1}}{\sqrt{n+2}} |x| \quad \overset{n \to \infty}{\longrightarrow} \quad 3|x| = L$$

Therefore:

$$|x| < \frac{1}{3} \quad \Longrightarrow \quad \text{converges}$$

$$|x| > \frac{1}{3} \quad \Longrightarrow \quad \text{diverges}$$

Preliminary interval: $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

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(2) Check endpoints:

Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(-\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$$

$\gg\gg$ diverges by LCT with $b_n = 1/\sqrt{n}$

Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(+\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

$\gg\gg$ converges by AST

Final interval of convergence: $x \in \left(-1/3, 1/3\right]$

Exercise - Radius and interval

Find the radius and interval of convergence of the following series:

(a) $\sum_{n=0}^{\infty} x^n$ (b) $\sum_{n=0}^{\infty} n!\, x^n$

Interval of convergence - further examples

Find the interval of convergence of the following series.

(a) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ (b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

Ratio of terms:

$$\frac{n+1}{3^{n+2}} \cdot \frac{3^{n+1}}{n}|x+2| \quad \gg\gg \quad \frac{n+1}{3n}|x+2| \quad \overset{n\to\infty}{\longrightarrow} \quad \frac{1}{3}|x+2| = L$$

This converges when $L < 1$ or $|x-(-2)| < 3$.

Therefore $R = 3$ and the preliminary interval is $x \in (-5, 1)$.

Check endpoints: $\sum \frac{n(-3)^n}{3^{n+1}}$ diverges and $\sum \frac{n(3)^n}{3^{n+1}}$ also diverges.

Final interval is $(-5, 1)$.

(b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Ratio of terms:

$$\left|\frac{a_{n+1}}{a_n}\right| \quad \gg\gg \quad \frac{\frac{1}{n+1}}{\frac{1}{n}}|4x+1|$$

$$\gg\gg \quad \frac{n}{n+1}|4x+1| \quad \overset{n\to\infty}{\longrightarrow} \quad |4x+1| = L$$

This converges when $L < 1$ or:

$$|4x+1| < 1 \quad \iff \quad |x+1/4| < 1/4 \quad \implies \quad \text{converges}$$

$$|4x+1| > 1 \quad \iff \quad |x+1/4| > 1/4 \quad \implies \quad \text{diverges}$$

Preliminary interval: $x \in (-1/2, 0)$

Check endpoints: $\sum \frac{(4\cdot\frac{-1}{2}+1)^n}{n}$ converges but $\sum \frac{1}{n}$ diverges.

Final interval of convergence: $[-1/2, 0)$