W11 - Notes

Power series as functions

Videos

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- Power series functions: Derivative/Antiderivative - Basics
- Power series functions: Derivative/Antiderivative - Interval of Convergence
- Power series functions: Derivative/Antiderivative - More examples
- Power series functions: Geometric Power Series

01 Theory

Given a numerical value for x within the interval of convergence of a power series, the series sum may be considered as the output $f(x)$ of a function f.

Many techniques from algebra and calculus can be applied to such power series functions.

Addition and Subtraction:

$$\begin{aligned} f &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ g &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ \hline f + g &= (a_0 + b_0) + (a_1 + b_1)\, x + (a_2 + b_2)\, x^2 + \cdots \end{aligned}$$

Summation notation:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \quad = \quad \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

Scaling:

$$cf \;=\; ca_0 + (ca_1)\, x + (ca_2)\, x^2 + \cdots$$

Summation notation:

$$c \sum_{n=0}^{\infty} a_n x^n \quad = \quad \sum_{n=0}^{\infty} (ca_n)\, x^n$$

Extra - Multiplication and composition

Multiplication:

$$f \cdot g = \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \cdot \left(b_0 + b_1 x + b_2 x^2 + \cdots\right)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\, x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\, x^2 + \cdots$$

For example, suppose that the geometric power series $f(x) = 1 + x + x^2 + x^3 + \cdots$ converges, so $|x| < 1$. Then we have for its square:

$$\begin{aligned} f \cdot f = f(x)^2 &= (1 + x + x^2 + \cdots) \cdot (1 + x + x^2 + \cdots) \\ &= 1 + (1+1)x + (1+1+1)x^2 + \cdots \\ &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &= \sum_{n=0}^{\infty} (n+1)x^n \end{aligned}$$

Composition:

$$f(-x) = 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$\begin{aligned} f(2x^3) &= 1 + 2x^3 + (2x^3)^2 + \cdots \\ &= 1 + 2x^3 + 4x^6 + 8x^9 + \cdots \end{aligned}$$

Assume:

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{n=0}^{\infty} a_n x^n$$

Then:

Differentiation:

$$\frac{df}{dx} = a_1 + (2a_2)\, x + (3a_3)\, x^2 + \cdots = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Antidifferentiation:

$$\int f(x)\, dx = C + a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

For example, for the geometric series we have:

$$f = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{df}{dx} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\int f\, dx = C + x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4 + \cdots$$

Do the series created with sums, products, derivatives etc., all converge? On what interval?

For the algebraic operations, the resulting power series will converge wherever both of the original series converge.

For calculus operations, the *radius* is preserved, but the *endpoints* are not necessarily:

Power series calculus - Radius preserved

If the power series $f(x)$ has radius of convergence R, then the power series $f'(x)$ and $\int f\, dx$ also have the same radius of convergence R.

⚠ Power series calculus - Endpoints not preserved

It is possible that a power series $f(x)$ converges at an endpoint a of its interval of convergence, yet f' and $\int f\,dx$ do *not* converge at a.

Extra - Proof of radius for derivative and integral series

Suppose $f(x)$ has radius of convergence R:

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| \quad \gg\gg \quad \left|\frac{a_{n+1}}{a_n}\right|\cdot|x| \longrightarrow \frac{1}{R}\cdot|x| \quad \text{as } n\to\infty$$

Consider now the *derivative* f' and its successive-term ratios:

$$\left|\frac{(n+1)a_{n+1}x^n}{na_n x^{n-1}}\right| = \left(\frac{n+1}{n}\right)\cdot\left|\frac{a_{n+1}}{a_n}\right|\cdot|x| \quad \overset{n\to\infty}{\longrightarrow} \quad 1\cdot\frac{1}{R}\cdot|x| = \frac{1}{R}\cdot|x|$$

Consider now the *antiderivative* $\int f\,dx$ and its successive-term ratios:

$$\left|\frac{\left(\frac{1}{n+1}\right)a_n x^{n+1}}{\left(\frac{1}{n}\right)a_{n-1}x^n}\right| = \left(\frac{n}{n+1}\right)\cdot\left|\frac{a_n}{a_{n-1}}\right|\cdot|x| \quad \overset{n\to\infty}{\longrightarrow} \quad 1\cdot\frac{1}{R}\cdot|x| = \frac{1}{R}\cdot|x|$$

In both these cases the ratio test provides that the series converges when $|x| < R$.

02 Illustration

Example - Geometric series: algebra meets calculus

Consider the geometric series as a power series functions:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Take the derivative of both sides of the *function*:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \gg\gg \quad \frac{1}{(1-x)^2} \quad \gg\gg \quad \left(\frac{1}{1-x}\right)^2$$

This means f satisfies the identity:

$$f' = f^2$$

Now compute the derivative of the *series*:

$$1 + x + x^2 + x^3 + \cdots \quad \overset{\frac{d}{dx}}{\gg\gg} \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

On the other hand, compute the square of the series:

$$\left(1 + x + x^2 + x^3 + \cdots\right)^2 \quad \gg\gg \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So we find that the *same relationship holds*, namely $f' = f^2$, for the closed formula and the series formula for this function.

Example - Manipulating geometric series: algebra

Find power series that represent the following functions:

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1+x^2}$ (c) $\dfrac{x^3}{x+2}$ (d) $\dfrac{3x}{2-5x}$

Solution

(a) $\dfrac{1}{1+x}$

Rewrite in format $\frac{1}{1-u}$:

$$\frac{1}{1+x} \quad \gg\gg \quad \frac{1}{1-(-x)}$$

Choose $u=-x$. Plug into geometric series:

$$\frac{1}{1-u} \;=\; 1+u+u^2+u^3+\cdots$$

$$\frac{1}{1-(-x)} \;=\; 1+(-x)+(-x)^2+(-x)^3+\cdots$$

$$\gg\gg \qquad 1-x+x^2-x^3+\cdots$$

Therefore:

$$\frac{1}{1+x} \;=\; 1-x+x^2-x^3+\cdots \;=\; \sum_{n=0}^{\infty}(-1)^n x^n$$

(b) $\dfrac{1}{1+x^2}$

Rewrite in format $\frac{1}{1-u}$:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Choose $u=-x^2$. Plug into geometric series:

$$\frac{1}{1-u} \;=\; 1+u+u^2+u^3+\cdots$$

$$\frac{1}{1-(-x^2)} \;=\; 1+(-x^2)+(-x^2)^2+(-x^2)^3+\cdots$$

$$\gg\gg \qquad 1-x^2+x^4-x^6+\cdots$$

Therefore:

$$\frac{1}{1+x^2} \;=\; 1-x^2+x^4-x^6+\cdots \;=\; \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

(c) $\dfrac{x^3}{x+2}$

Rewrite in format $Ax^3\cdot\frac{1}{1-u}$:

$$\frac{x^3}{x+2} \quad \ggg \quad x^3 \cdot \frac{1}{2+x} \quad \ggg \quad x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$\ggg \quad \frac{1}{2}x^3 \cdot \frac{1}{1+\frac{x}{2}} \quad \ggg \quad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

Choose $u = -\frac{x}{2}$. Here $Ax^3 = \frac{1}{2}x^3$. Plug into geometric series:

$$\frac{1}{1-u} \;=\; 1 + u + u^2 + u^3 + \cdots$$

$$\frac{1}{1-\left(-\frac{x}{2}\right)} \;=\; \sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n \;=\; 1 + \left(-\tfrac{x}{2}\right) + \left(-\tfrac{x}{2}\right)^2 + \left(-\tfrac{x}{2}\right)^3 + \cdots$$

$$\ggg \quad \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}x^n \;=\; 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

Therefore:

$$\frac{x^3}{x+2} \quad \ggg \quad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

$$\ggg \quad \frac{1}{2}x^3 \left(\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}x^n\right)$$

$$\ggg \quad \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{n+1}}x^{n+3} \;=\; \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

(d) $\dfrac{3x}{2-5x}$

Rewrite in format $Ax \cdot \frac{1}{1-u}$:

$$\frac{3x}{2-5x} \quad \ggg \quad 3x \cdot \frac{1}{2-5x}$$

$$\ggg \quad 3x \cdot \frac{1}{2\left(1-\frac{5x}{2}\right)} \quad \ggg \quad \frac{3}{2}x \cdot \frac{1}{1-\frac{5}{2}x}$$

Choose $u = \frac{5}{2}x$. Here $Ax = \frac{3}{2}x$. Plug into geometric series:

$$\frac{1}{1-u} \;=\; 1 + u + u^2 + u^3 + \cdots$$

$$\frac{1}{1-\left(\frac{5}{2}x\right)} \;=\; \sum_{n=0}^{\infty}\left(\frac{5}{2}x\right)^n \;=\; 1 + \left(\tfrac{5x}{2}\right) + \left(\tfrac{5x}{2}\right)^2 + \left(\tfrac{5x}{2}\right)^3 + \cdots$$

$$\ggg \quad \sum_{n=0}^{\infty}\frac{5^n}{2^n}x^n \;=\; 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

Therefore:

$$\frac{3x}{2-5x} \quad \gg\gg \quad \frac{3}{2}x \cdot \frac{1}{1-\frac{5}{2}x}$$

$$\gg\gg \quad \frac{3}{2}x\left(\sum_{n=0}^{\infty}\frac{5^n}{2^n}x^n\right)$$

$$\gg\gg \quad \sum_{n=0}^{\infty}\frac{3\cdot 5^n}{2^{n+1}}x^{n+1} = \frac{3}{2}x+\frac{15}{4}x^2+\frac{75}{8}x^3+\frac{375}{16}x^4+\cdots$$

Example - Manipulating geometric series: calculus

Find a power series that represents $\ln(1+x)$.

Solution

Differentiate to obtain similarity to geometric sum formula:

$$\frac{d}{dx}\ln(1+x) \quad \gg\gg \quad \frac{1}{1+x}$$

$$\gg\gg \quad \frac{1}{1-(-x)} \quad \gg\gg \quad 1-x+x^2-x^3+x^4-\cdots$$

Integrate series to find original function:

$$\int \frac{1}{1-(-x)}\,dx = \int 1-x+x^2-x^3+x^4-\cdots\,dx$$

$$\gg\gg \quad \ln(1+x) = D+x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

Use known point to solve for D:

$$\ln(1+0) = D+0+0+\cdots \qquad \gg\gg \qquad 0=D$$

$$\gg\gg \quad \ln(1+x) = x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots$$

Example - Recognizing and manipulating geometric series: Part I

(a) Evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. (Hint: consider the series of $\ln(1-x)$.)

(b) Find a series approximation for $\ln(2/3)$.

Solution

(a)

(1) Follow hint, study series of $\ln(1-x)$:

Notice:

$$\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x} \quad \gg\gg \quad -(1+x+x^2+x^3+\cdots)$$

Integrate the series:

$$\int \frac{-1}{1-x}\,dx \;=\; \int -1-x-x^2-\cdots\,dx$$

$$\gg\gg \quad \ln(1-x) \;=\; C-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots$$

Solve for C using $\ln(1-0)=0$ which (plugging above) implies $0=C-0-0-\cdots$ and thus $C=0$. So:

$$\ln(1-x) \;=\; -x-\frac{1}{2}x^2-\frac{1}{3}x^3-\cdots \quad = \quad \sum_{n=1}^{\infty}-\frac{1}{n}x^n$$

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(2) Relate to the given series:

Notice that $x^n=(-1)^n$ if we set $x=-1$.
Also, $-(-1)^n=(-1)^{n-1}$.
Therefore:

$$\ln(1-(-1)) \quad \gg\gg \quad \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$$

So the answer is $\ln 2$.

(b) Find a series approximation for $\ln(2/3)$:

Observe that $\ln(2/3)=\ln(1-1/3)$.

Plug $x=1/3$ into the series: $\ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$

$$\ln(1-1/3) \quad \gg\gg \quad -1/3-\frac{(1/3)^2}{2}-\frac{(1/3)^3}{3}-\cdots$$

$$\gg\gg \quad -\frac{1}{3}-\frac{1}{3^2\cdot 2}-\frac{1}{3^3\cdot 3}-\cdots$$

$$\gg\gg \quad \sum_{n=1}^{\infty}-\frac{1}{n\,3^n}$$

Example - Recognizing and manipulating geometric series: Part II

(a) Find a series representing $\tan^{-1}(x)$ using differentiation.

(b) Find a series representing $\displaystyle\int \frac{dx}{1+x^4}$.

Solution

(a)

Notice that $\frac{d}{dx}\tan^{-1}(x)=\frac{1}{1+x^2}$.

What is the series for $\frac{1}{1+x^2}$?

Let $u = -x^2$:

$$\frac{1}{1+x^2} \quad \gg\gg \quad \frac{1}{1-u} = 1 + u + u^2 + \cdots$$

$$\gg\gg \quad 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Now integrate this by terms:

$$\int \frac{1}{1+x^2}\,dx \;=\; \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots\,dx$$

$$\gg\gg \quad C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Conclude:

$$\tan^{-1}(x) \;=\; C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Plug in 0 to solve for C:

$$\tan^{-1}(0) \;=\; C + 0 + \cdots + 0 \quad \gg\gg \quad C = 0$$

Final answer:

$$\tan^{-1}(x) \;=\; x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

(b)

Rewrite integrand in format of geometric series sum:

$$\frac{1}{1+x^4} \quad \gg\gg \quad \frac{1}{1-(-x^4)} \quad \gg\gg \quad \frac{1}{1-u}, \quad u = -x^4$$

Therefore:

$$\frac{1}{1-u} \;=\; 1 + u + u^2 + u^3 + \cdots$$

$$\gg\gg \quad 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

Integrate the series by terms to obtain the answer:

$$\int 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots\,dx$$

$$\gg\gg \quad C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots$$

Taylor and Maclaurin series

Videos

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- Maclaurin series: $f(x) = \frac{1}{(1-x)^2}$
- Maclaurin series: $f(x) = e^x$
- Maclaurin series: $f(x) = \sin x,\ \cos x,\ \tan x$
- Taylor series: $f(x) = \ln x$ at $x = 1$

03 Theory

Suppose that we have a power series function:

$$f(x) \ = \ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Consider the *successive derivatives* of f:

$$\begin{array}{rcccccccccccc}
f(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + & \cdots \\
f'(x) & = & 0 & + & a_1 & + & 2 \cdot a_2 x^1 & + & 3 \cdot a_3 x^2 & + & 4 \cdot a_4 x^3 & + & \cdots \\
f''(x) & = & 0 & + & 0 & + & 2 \cdot a_2 & + & 3 \cdot 2 \cdot a_3 x^1 & + & 4 \cdot 3 \cdot a_4 x^2 & + & \cdots \\
f'''(x) & = & 0 & + & 0 & + & 0 & + & 3 \cdot 2 \cdot 1 \cdot a_3 & + & 4 \cdot 3 \cdot 2 \cdot a_4 x^1 & + & \ldots \\
\vdots & & & & \vdots & & & & \vdots & & & & \\
f^{(n)}(x) & = & 0 & + & 0 & + & 0 & + & 0 & + & \cdots + n! \cdot a_n & + & \cdots
\end{array}$$

When these functions are evaluated at $x = 0$, all terms with a positive x-power become zero:

$$\begin{array}{rcccc}
f(0) & = & a_0 & = & a_0 \\
f'(0) & = & a_1 & = & a_1 \\
f''(0) & = & 2 \cdot a_2 & = & 2! \cdot a_2 \\
f'''(0) & = & 3 \cdot 2 \cdot a_3 & = & 3! \cdot a_3 \\
\vdots & = & \vdots & = & \vdots \\
f^{(n)}(0) & = & n \cdot (n-1) \cdots 2 \cdot 1 \cdot a_n & = & n! \cdot a_n
\end{array}$$

This last formula is the basis for Taylor and Maclaurin series:

Power series: Derivative-Coefficient Identity

$$f^{(n)}(0) \ = \ n! \cdot a_n$$

This identity holds for a power series function $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ which has a nonzero radius of convergence.

We can apply the identity in both directions:

- Know $f(x)$? $\rightsquigarrow$ Calculate a_n for any n.
- Know a_n? $\rightsquigarrow$ Calculate $f^{(n)}(0)$ for large n. (Faster than differentiating.)

Many functions can be 'expressed' or 'represented' near $x = c$ (i.e. for small enough $|x - c|$) as convergent power series. (This is true for almost all the functions encountered in pre-calculus and calculus.)

Such a power series representation is called a **Taylor series**.
When $c = 0$, the Taylor series is also called the **Maclaurin series**.

One power series representation we have already studied:

$$\frac{1}{1-x} \ = \ 1 + x + x^2 + x^3 + \cdots$$

Whenever a function has a power series (Taylor or Maclaurin), the Derivative-Coefficient Identity may be applied to *calculate the coefficients* of that series.

Conversely, sometimes a series can be interpreted as an *evaluated power series* coming from $x = c$ for some c. If the closed form function format can be obtained for this power series, the *total sum of the original series may be discovered* by putting $x = c$ in the argument of the function.

04 Illustration

Example - Maclaurin series of e^x

What is the Maclaurin series of $f(x) = e^x$?

Solution

Using $\dfrac{d}{dx}e^x = e^x$ repeatedly, we see that $f^{(n)}(x) = e^x$ for all n.

So $f^{(n)}(0) = e^0 = 1$ for all n. Therefore $a_n = \dfrac{1}{n!}$ for all n by the Derivative-Coefficient Identity:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example - Maclaurin series of $\cos x$

Find the Maclaurin series representation of $\cos x$.

Solution

Use the Derivative-Coefficient Identity to solve for the coefficients:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/24$
5	$-\sin x$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By studying this pattern, we find the series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Maclaurin series from other Maclaurin series

(a) Find the Maclaurin series of $\sin x$ using the Maclaurin series of $\cos x$.

(b) Find the Maclaurin series of $f(x) = x^2 e^{-5x}$ using the Maclaurin series of e^x.

(c) Using (b), find the value of $f^{(22)}(0)$.

Solution

(a)

Remember that $\frac{d}{dx}\cos x = -\sin x$. Let us differentiate the cosine series by terms:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \overset{\frac{d}{dx}}{\gg\gg} \quad 0 - 2\frac{x^1}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$

$$\gg\gg \quad -\frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

Take negative to get:

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(b)

$$e^u \;=\; 1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

Set $u = -5x$:

$$e^{-5x} \;=\; 1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots \;=\; \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{n!}\, x^n$$

Multiply all terms by x^2:

$$x^2 e^{-5x} \quad \gg\gg \quad x^2\left(1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots\right)$$

$$\gg\gg \quad x^2 - 5x^3 + \frac{25}{2}x^4 - \frac{125}{3!}x^5 + \cdots$$

$$\gg\gg \quad \sum_{n=0}^{\infty} (-1)^n \frac{5^n}{n!}\, x^{n+2}$$

(c)

For any series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

we have:

$$f^{(n)}(0) = n! \cdot a_n$$

We can use this to compute a_{22}. From the series formula:

$$\sum_{n=0}^{\infty}(-1)^n\frac{5^n}{n!}\,x^{n+2}$$

we see that:

$$a_{n+2}=(-1)^n\frac{5^n}{n!}$$

> **⚠ Power, NOT term number**
>
> The coefficient with a_{n+2} corresponds to the term having x^{n+2}, *not necessarily* the $(n+2)^{\text{th}}$ term of the series.

Therefore:

$$a_{22} \;=\; (-1)^{20}\frac{5^{20}}{20!} \quad\gg\gg\quad 5^{20}\frac{1}{20!}$$

$$f^{(22)}(0) \;=\; 22!\cdot a_{22} \quad\gg\gg\quad 5^{20}\cdot\frac{22!}{20!} \quad\gg\gg\quad 5^{20}\cdot 22\cdot 21$$

Computing a Taylor series

Find the first five terms of the Taylor series of $f(x)=\sqrt{x+1}$ centered at $c=3$.

Solution

A Taylor series is just a Maclaurin series centered at a nonzero number.

General format of a Taylor series:

$$f(x)=a_0+a_1(x-c)+a_2(x-c)^2+a_3(x-c)^3+\cdots$$

The coefficients satisfy $f^{(n)}(c)=n!\,a_n$.

Find the coefficients by computing the derivatives and evaluating at $x=3$:

$$\begin{aligned}
f(x)&=(x+1)^{1/2}, & f(3)&=2\\
f'(x)&=\frac{1}{2}(x+1)^{-1/2}, & f'(3)&=\frac{1}{4}\\
f''(x)&=-\frac{1}{4}(x+1)^{-3/2}, & f''(3)&=-\frac{1}{32}\\
f'''(x)&=\frac{3}{8}(x+1)^{-5/2}, & f'''(3)&=\frac{3}{256}\\
f^{(4)}(x)&=-\frac{15}{16}(x+1)^{-7/2}, & f^{(4)}(3)&=-\frac{15}{2048}
\end{aligned}$$

The first terms of the series:

$$f(x) \;=\; \sqrt{x+1}$$

$$=\; 2+\frac{1}{4}(x-3)-\frac{1}{64}(x-3)^2+\frac{1}{512}(x-3)^3-\frac{5}{16,384}(x-3)^4+\cdots$$

05 Theory

⚠ Study these!

- Memorize all of these series!
- Recognize all of these series!
- Recognize all of these summation formulas!

$$\begin{aligned}
\frac{1}{1-x} = 1 + x + x^2 + \cdots &= \sum_{n=0}^{\infty} x^n, \quad R = 1, \quad \text{interval: } (-1, 1) \\
\ln(1-x) = -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \cdots &= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \quad R = 1, \quad \text{interval: } [-1, 1) \\
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad R = 1, \quad \text{interval: } [-1, 1] \\
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = \infty \\
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad R = \infty \\
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad R = \infty
\end{aligned}$$

Applications of Taylor series

Videos

Videos, Math Dr. Bob

- Approximating with Maclaurin polynomials: $f(x) = \ln(1-x)$ to find $\ln(1.1)$
- Approximating with Taylor polynomials: $f(x) = \frac{1}{x+1}$ at $x = 1$ to find $1/2.1$

06 Theory reminder

Linear approximation is the technique of approximating a specific value of a function, say $f(x_1)$, at a point x_1 that is close to another point x_0 where we *know* the exact value $f(x_0)$. We write Δx for $x_1 - x_0$, and $y_0 = f(x_0)$, and $y_1 = f(x_1)$. Then we write $dy = f'(x_0) \cdot \Delta x$ and use the fact that:

$$y_1 \approx y_0 + dy = y_0 + f'(x_0) \cdot \Delta x$$

☰ Computing a linear approximation

For example, to approximate the value of $\sqrt{4.01}$, set $f(x) = \sqrt{x}$, set $x_0 = 4$ and $y_0 = 2$, and set $x_1 = 4.01$ so $\Delta x = 0.01$.

Then compute: $f'(x) = \frac{1}{2\sqrt{x}}$
So $f'(x_0) = 1/4$.

Finally:

$$y_1 \approx y_0 + f'(x_0) \cdot \Delta x \qquad \gg\gg \qquad y_1 \approx 2 + \frac{1}{4} \cdot 0.01 = 2.0025$$

Now recall the **linearization** of a function, which is itself another function:

Given a function $f(x)$, the linearization $L(x)$ at the basepoint $x = c$ is the functional form of the tangent line, the line passing through $(x_0, y_0) = (c, f(c))$ with slope $m = f'(c)$:

$$y = y_0 + m(x - x_0) \qquad \text{(point-slope eqn. of line)}$$

$$\gg\gg \quad L(x) = f(c) + f'(c)(x - c) \qquad \text{(linearization fcn.)}$$

The graph of this linearization $L(x)$ is the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$.

The linearization $L(x)$ may be used as a replacement for $f(x)$ for values of x near c. The closer x is to c, the more accurate the approximation $L(x)$ is for $f(x)$.

Computing a linearization

We set $f(x) = \sqrt{x}$, and we let $c = 4$.

We compute $f(c) = 2$, and $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(c) = \frac{1}{4}$.

Plug everything in to find $L(x)$:

$$L(x) = f(c) + f'(c)(x - c) \quad \gg\gg \quad L(x) = 2 + \frac{1}{4}(x - 4)$$

Now approximate $f(4.01) \approx L(4.01)$:

$$L(4.01) = 2 + \frac{1}{4}(4.01 - 4) = 2.0025$$

07 Theory

Taylor polynomials

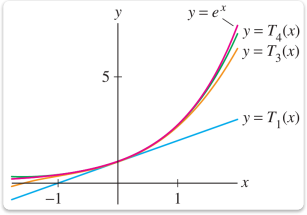
The **Taylor polynomials** $T_N(x)$ of a function $f(x)$ are the partial sums of the Taylor series of $f(x)$:

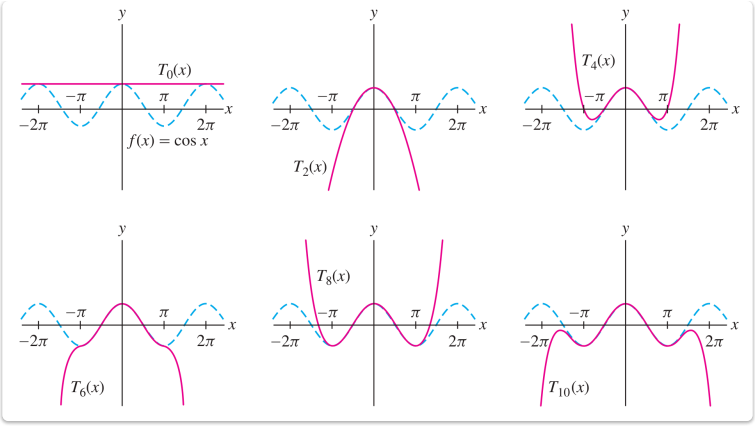
$$T_N(x) = \sum_{i=0}^{N} \frac{f^{(i)}(c)}{i!}(x - c)^i$$

$$= f(c) + \frac{f'(c)}{1!}(x - c) + \cdots + \frac{f^{(N)}(c)}{N!}(x - c)^N$$

These polynomials are *generalizations of linearization*.
Specifically, $f(c) = T_0(x)$, and $L(x) = T_1(x)$.

The Taylor series $T_n(x)$ is a better approximation of $f(x)$ than $T_i(x)$ for any $i < n$.





Facts about Taylor series

The series $T_n(x)$ has the same derivatives as $f(x)$ at the point $x = c$. This fact can be verified by visual inspection of the series: apply the power rule and chain rule, then plug in $x = c$ and all factors left with $(x - c)$ will become zero.

The difference $f(x) - T_n(x)$ vanishes to order n at $x = c$:

$$\begin{aligned} f(x) - T_n(x) &= \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + \cdots \\ &= (x-c)^n \left(\frac{f^{(n)}(c)}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c) + \cdots \right) \end{aligned}$$

The factor $(x - c)^n$ drives the whole function to zero with order n as $x \to c$.

If we only considered orders up to n, we might say that $f(x)$ and $T_n(x)$ are the same near c.

08 Illustration

Taylor polynomial approximations

Let $f(x) = \sin x$ and let $T_n(x)$ be the Taylor polynomials expanded around $c = 0$.

By considering the alternating series error bound, find the first n for which $T_n(0.02)$ must have error less than 10^{-6}.

Solution

Write the Maclaurin series of $\sin x$ because we are expanding around $c = 0$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

This series is alternating when we insert $x = 0.02$, so the AST error bound formula applies ("Next Term Bound"):

$$|E_n| \leq a_{n+1}$$

Find smallest n such that $a_{n+1} \leq 10^{-6}$, and then we know:

$$|E_n| \leq a_{n+1} \leq 10^{-6} \quad \gg\gg \quad |E_n| \leq 10^{-6}$$

Plug $x = 0.02$ in the series for $\sin x$:

$$a_{2n+1} = \frac{(0.02)^{2n+1}}{(2n+1)!}$$

Solve for the first time $a_{2n+1} \leq 10^{-6}$ by listing the values:

$$\frac{0.02^1}{1!} = 0.02, \qquad \frac{0.02^3}{3!} \approx 1.33 \times 10^{-6},$$

$$\frac{0.02^5}{5!} \approx 2.67 \times 10^{-11}, \quad \ldots$$

The first time a_{2n+1} is below 10^{-6} happens when $2n + 1 = 5$.

This is NOT the same n as in T_n. That n is the highest power of x allowed.

The sum of prior terms is $T_4(0.02)$.

Since $T_4(x) = T_3(x)$ because there is no x^4 term, the final answer is $n = 3$.

Taylor polynomials to approximate a definite integral

Approximate $\displaystyle\int_0^{0.3} e^{-x^2}\,dx$ using a Taylor polynomial with an error no greater than 10^{-5}.

Solution

Plug $u = -x^2$ into the series of e^u:

$$e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$$

$$\gg\gg \quad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} \quad \gg\gg \quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{2n}$$

Find an antiderivative by terms:

$$\int \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{2n}\,dx$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{x^{2n+1}}{2n+1}$$

$$\gg\gg \quad x - \frac{1}{1!(3)} x^3 + \frac{1}{2!(5)} x^5 - \frac{1}{3!(7)} x^7 + \cdots$$

Plug in bounds for definite integral:

$$\int_0^{0.3} \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{x^{2n+1}}{2n+1} \Bigg|_0^{0.3}$$

$$\gg\gg \quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{(0.3)^{2n+1}}{2n+1}$$

$$\gg\gg \quad 0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{10} - \frac{(0.3)^7}{42} + \cdots$$

Notice alternating series pattern. Apply error bound formula, “Next Term Bound”:

$$\frac{(0.3)^5}{10} \approx 2.43 \times 10^{-4}, \qquad \frac{(0.3)^7}{42} \approx 5.21 \times 10^{-6}$$

So we can guarantee an error less than 5.21×10^{-6} by summing the first terms through $\frac{(0.3)^5}{10}$:

$$0.3 - \frac{(0.3)^3}{3} + \frac{(0.3)^5}{10} \;=\; \boxed{0.291243}$$