W13 - Notes

More calculus with parametric curves

01 Theory - Arclength

Arclength formula

The **arclength** of a parametric curve with coordinate functions $x(t)$ and $y(t)$ is:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

This formula assumes the curve is traversed one time as t increases from a to b.

Counts total traversal

This formula applies when the curve image is traversed *one time* by the moving point.

Sometimes a parametric curve traverses its image with repetitions. The arclength formula would add length from each repetition!

Extra - Derivation of arclength formula

The arclength of a parametric curve is calculated by integrating the infinitesimal arc element:

$$ds = \sqrt{dx^2 + dy^2}$$

$$L = \int_a^b ds$$

In order to integrate ds in the t variable, as we must for parametric curves, we convert ds to a function of t:

$$ds = \sqrt{dx^2 + dy^2} \quad \gg\gg \quad \sqrt{\frac{1}{dt^2} \cdot (dx^2 + dy^2) \cdot dt^2}$$

$$\gg\gg \quad \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}} \cdot \sqrt{dt^2} \quad \gg\gg \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\gg\gg \quad ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

So we obtain $ds = \sqrt{(x')^2 + (y')^2}\, dt$ and the arclength formula follows from this:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

02 Illustration

Example - Perimeter of a circles

(1) The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 \;=\; (-R\sin t)^2 + (R\cos t)^2$$

$$\gg\gg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \gg\gg \qquad R^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = R\, dt$$

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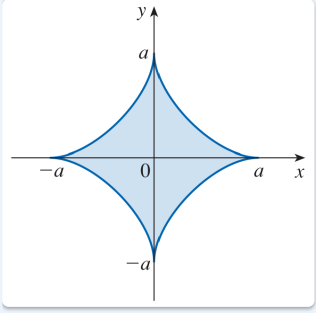
(2) Integrate around the circle:

$$\text{Perimeter} \;=\; \int_0^{2\pi} ds \quad \gg\gg \quad \int_0^{2\pi} R\, dt$$

$$\gg\gg \quad Rt\Big|_0^{2\pi} = 2\pi R$$

Example - Perimeter of an asteroid

Find the perimeter length of the 'asteroid' given parametrically by $(x, y) = \left(a\cos^3\theta,\, a\sin^3\theta\right)$ for $a = 2$.



Solution

(1) Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$(x', y') = \left(3a\cos^2\theta\sin\theta,\, 3a\sin^2\theta\cos\theta\right)$$

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(2) Compute the infinitesimal arc element.

$$(x')^2 + (y')^2 \quad \gg\gg \quad 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$\gg\gg \quad 9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$\gg\gg \quad 9a^2\sin^2\theta\cos^2\theta$$

Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + y')^2}\,d\theta$$

$$\gg\gg \quad \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta$$

$$\gg\gg \quad ds = 3a\left|\sin\theta\cos\theta\right|d\theta$$

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(3) Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a\sin\theta\cos\theta$ is *positive*. So we can drop the absolute value and integrate directly.

> **⚠ Absolute values matter!**
>
> If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a\sin\theta\cos\theta$ really does change sign.
>
> To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

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(4) Integrate the arc element:

$$\int_0^{\pi/2} ds \quad \gg\gg \quad \int_0^{\pi/2} 3a\sin\theta\cos\theta\,d\theta$$

$$\gg\gg \quad 3a\int_{u=0}^{1} u\,du \qquad (u = \sin\theta)$$

$$\gg\gg \quad 3a\frac{u^2}{2}\bigg|_0^1 \quad \gg\gg \quad \frac{3a}{2}$$

Finally, multiply by 4 to get the total perimeter: $L = 6a$

03 Theory - Distance, speed

Distance function

The **distance function** $s(t)$ returns the total distance traveled by the particle from a chosen starting time t_0 up to the (input) time t:

$$s(t) \;=\; \int_{t_0}^{t} ds \quad = \quad \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

We need the dummy variable u so that the integration process does not conflict with t in the upper bound.

Speed function

The **speed** of a moving particle is the *rate of change of distance*:

$$v(t) \;=\; s'(t) \quad = \quad \sqrt{x'(t)^2 + y'(t)^2}$$

This formula can be explained in either of two ways:

1. Apply the Fundamental Theorem of Calculus to the integral formula for $s(t)$.
2. Consider $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ for a small change dt: so the *rate of change* of arclength is $\frac{ds}{dt}$, in other words $s'(t)$.

04 Illustration

Example - Speed, distance, displacement

The parametric curve $\left(t, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)

Compute *derivatives*:

$$(x',\, y') = \left(1,\, t^{1/2}\right)$$

Now compute the *speed*:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t \;\gg\gg\quad v(t) = \sqrt{(x')^2 + (y')^2} = \boxed{\sqrt{1+t}}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

Substitute $w = 1 + u$ and $dw = du$. New bounds are 1 and $1 + t$. Calculate:

$$\gg\gg \quad \int_{1}^{1+t} \sqrt{w}\, dw$$

$$\gg\gg \quad \frac{2}{3} w^{3/2} \Big|_{1}^{1+t} \quad \gg\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\gg \quad \frac{2}{3}(27 - 1) \quad \gg\gg \quad \boxed{\frac{52}{3}}$$

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Now compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \quad \gg\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=0} \quad \gg\gg \quad (0, 0)$$

For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \gg\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=8}$$

$$\gg\gg \quad \left(8, \frac{2}{3}8^{3/2}\right) \quad \gg\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Insert $(0, 0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\gg\gg \quad \sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\gg \quad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\gg \quad \boxed{\frac{\sqrt{2624}}{3}}$$

05 Theory - Surface area of revolutions

Surface area of a surface of revolution: thin bands

Suppose a parametric curve $\big(x(t),\, y(t)\big)$ is revolved around the x-axis or the y-axis.

The surface area is:

$$A \;=\; \int_a^b 2\pi R(t)\,\sqrt{(x')^2 + (y')^2}\,dt$$

The radius $R(t)$ should be the distance to the axis:

$$\begin{aligned} R(t) &= y(t) \qquad \text{revolution about } x\text{-axis} \\ R(t) &= x(t) \qquad \text{revolution about } y\text{-axis} \end{aligned}$$

This formulas adds the areas of thin bands, but the bands are demarcated using parametric functions instead of input values of a graphed function.

The formula assumes that the curve is traversed one time as t increases from a to b.

06 Illustration

⁝☰ Example - Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

Parametrization of the unit upper semicircle:

$$(x, y) = (\cos t,\, \sin t)$$

$$\gg\gg \quad (x', y') = (-\sin t,\, \cos t)$$

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\,dt$$

$$\gg\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt \quad \gg\gg \quad dt$$

Now for $R(t)$ we choose $R(t) = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^{\pi} 2\pi \sin t\,dt \quad \gg\gg \quad -2\pi\cos t\Big|_0^{\pi} \quad \gg\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

⁝☰ Example - Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3,\ t^2-1)$ for $0 \le t \le 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \gg\gg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \gg\gg \quad \sqrt{9t^4 + 4t^2}$$

$$\gg\gg \quad \sqrt{t^2(9t^2+4)} \quad \gg\gg \quad t\sqrt{9t^2+4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \gg\gg \quad \int_0^1 2\pi(t^2-1)t\sqrt{9t^2+4}\, dt$$

Polar curves

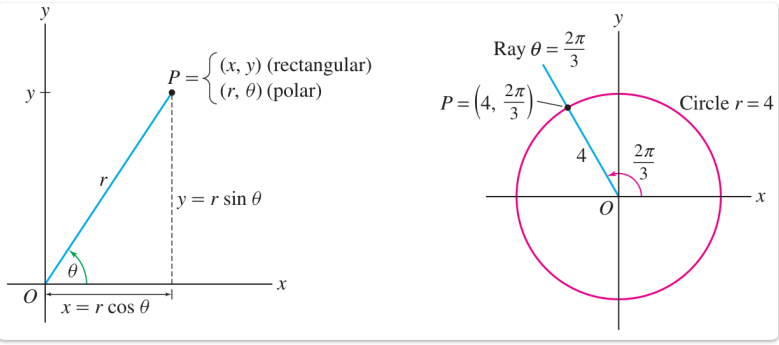
Videos

Videos, Organic Chemistry Tutor

- Polar coordinates intro
- Graphing polar curves

07 Theory - Polar points, polar curves

Polar coordinates are pairs of numbers (r, θ) which identify points in the plane in terms of *distance to origin* and *angle from $+x$-axis*:



Converting Polar ↔ Cartesian

Polar → Cartesian	Cartesian → Polar
$x = r\cos\theta$	$r = \sqrt{x^2+y^2}$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x} \quad (x \neq 0)$

Polar coordinates have *many redundancies*: unlike Cartesian which are unique!

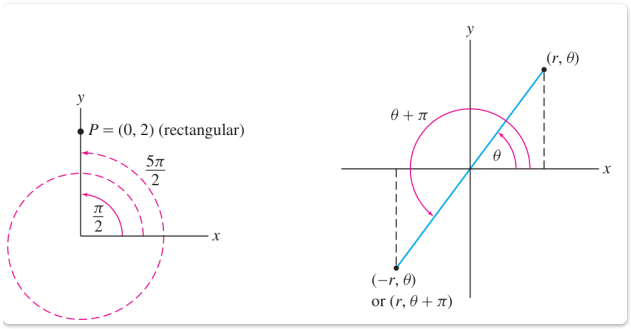
- For example: $(r, \theta) = (r, \theta + 2\pi)$
  - And therefore also $(r, \theta) = (r, \theta - 2\pi)$ (negative θ can happen)
- For example: $(-r, \theta) = (r, \theta + \pi)$ for every r, θ
- For example: $(0, \theta) = (0, 0)$ for any θ

Polar coordinates *cannot be added*: they are not vector components!

- For example $(5, \pi/3) + (2, \pi/6) \neq (7, \pi/2)$
- Whereas Cartesian coordinates can be added: $(1, 4) + (2, -2) = (3, 2)$

⚠ **The transition formulas** Cartesian $\to$ Polar **require careful choice of** θ**.**

- The standard definition of $\tan^{-1}\left(\frac{y}{x}\right)$ sometimes gives *wrong* θ
  - This is because it uses the restricted domain $\theta \in (-\pi/2, \pi/2)$; the polar interpretation is: only points in Quadrant I and Quadrant IV (SAFE QUADRANTS)
- Therefore: *check signs* of x and y to see *which quadrant*, maybe need π-correction!
  - Quadrant I or IV: polar angle is $\tan^{-1}\left(\frac{y}{x}\right)$
  - Quadrant II or III: polar angle is $\tan^{-1}\left(\frac{y}{x}\right) + \pi$



Equations (as well as points) can also be converted to polar.

For Cartesian $\to$ Polar, look for cancellation from $\cos^2\theta + \sin^2\theta = 1$.

For Polar $\to$ Cartesian, try to keep θ inside of trig functions.

- For example:

$$r = \sin^2\theta \qquad \gg\gg \qquad \sqrt{x^2 + y^2} = \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2$$

08 Illustration

Example - Converting to polar: π-correction

Compute the polar coordinates of $\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \gg\gg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \gg\gg \quad -\pi/3$$

This angle is in Quadrant IV. We *add* π to get the polar angle in Quadrant II:

$$\theta = \pi - \pi/3 \quad \gg\gg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r, \theta) = (1, 2\pi/3)$.

For $\left(+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV. (No extra π needed.)

Next compute:

$$\tan^{-1}\left(\frac{-\sqrt{2}/2}{+\sqrt{2}/2}\right) \quad \gg\gg \quad \tan^{-1}(-1) \quad \gg\gg \quad -\pi/4$$

So the point in polar is $(1, -\pi/4)$.

Example - Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y-3)^2 \;=\; 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \quad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \quad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

$$\gg\gg \quad r^2 - 6r\sin\theta = 0 \quad \gg\gg \quad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.