W14 - Notes

More polar curves

01 Theory - Polar limaçons

To draw the polar graph of some function, it can help to first draw the Cartesian graph of the function. (In other words, set $y = r$ and $x = \theta$, and draw the usual graph.) By tracing through the points on the Cartesian graph, one can visualize the trajectory of the polar graph.

This Cartesian graph may be called a **graphing tool** for the polar graph.

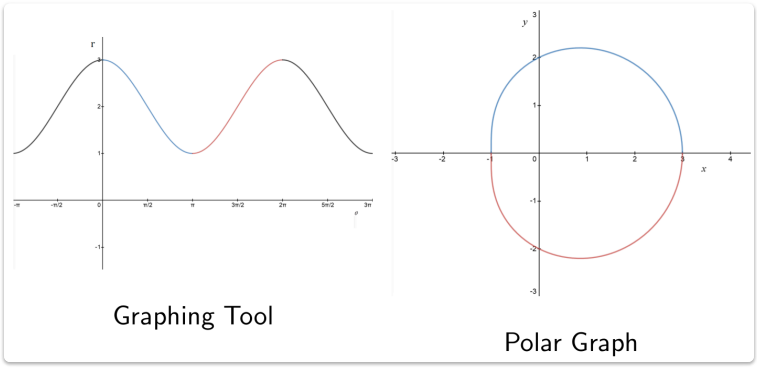
A limaçon is the polar graph of $r(\theta) = a + b\cos\theta$.

The *shape* of a limaçon is determined by the value of $c = \dfrac{b}{a}$. Any limaçon can be rescaled to have this form:

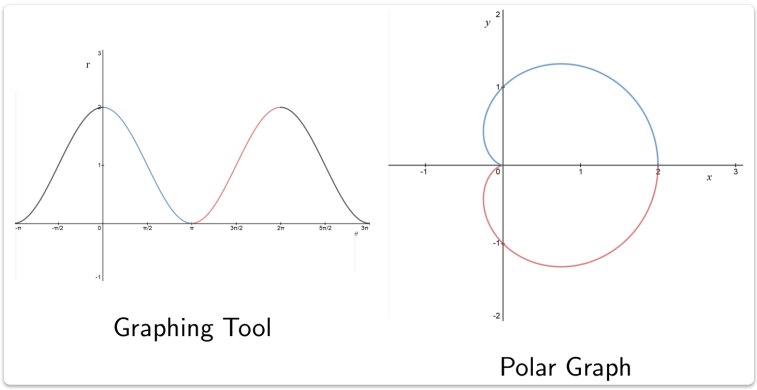
$$r = 1 + c\cos\theta$$

$c = 0$: Limaçon satisfying $r(\theta) = 1$: unit circle.

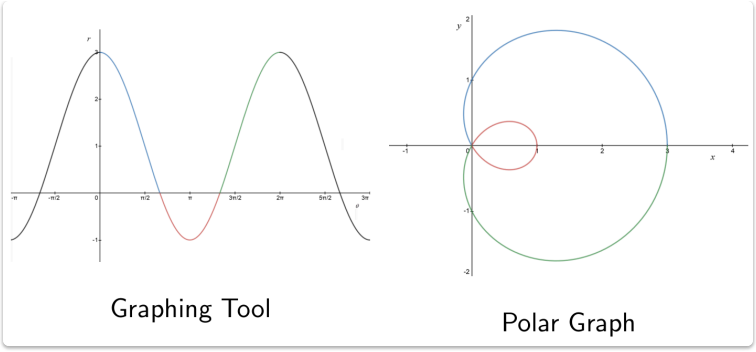
$c = 0.5$: Limaçon satisfying $r(\theta) = 2 + \cos\theta$: 'outer loop' circle with 'flat spot', not quite a 'dimple':



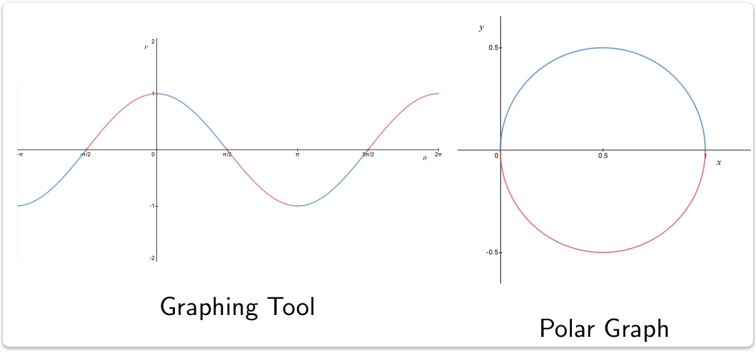
$c = 1$: Limaçon satisfying $r(\theta) = 1 + \cos\theta$: 'cardioid' = 'outer loop' circle with 'dimple' that creates a cusp:



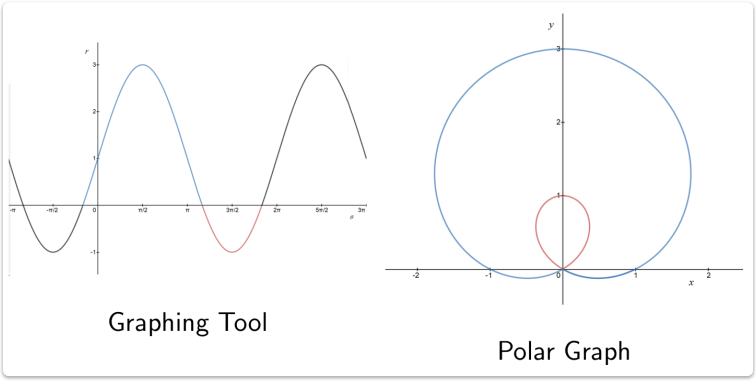
$c = 2$: Limaçon satisfying $r(\theta) = 1 + 2\cos\theta$: 'dimple' pushes past cusp to create 'inner loop':



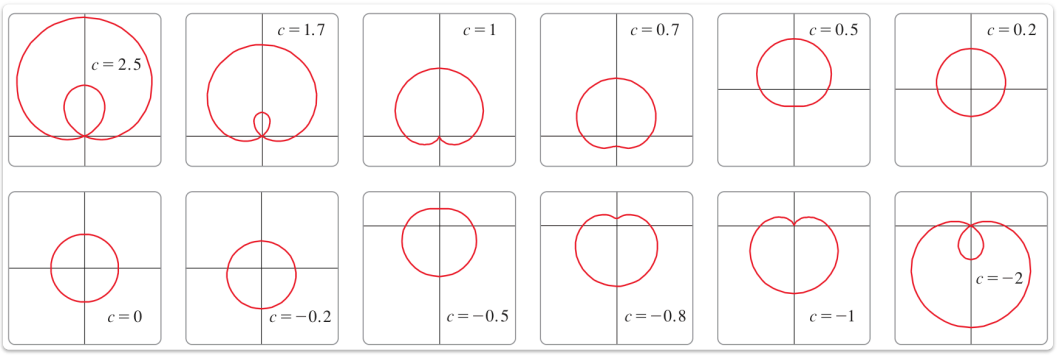
$c = \infty$: Limaçon satisfying $r(\theta) = \cos\theta$: 'inner loop' only, no outer loop exists:



$c = 2$: Limaçon satisfying $r(\theta) = 1 + 2\sin\theta$: 'inner loop' and 'outer loop' and rotated $\circlearrowleft 90°$:



Transitions between limaçon types, $r(\theta) = 1 + c\sin\theta$:



Notice the transition points at $|c| = 0.5$ and $|c| = 1$:

The *flat spot* occurs when $c = \pm 0.5$

- Smaller c gives *convex shape*

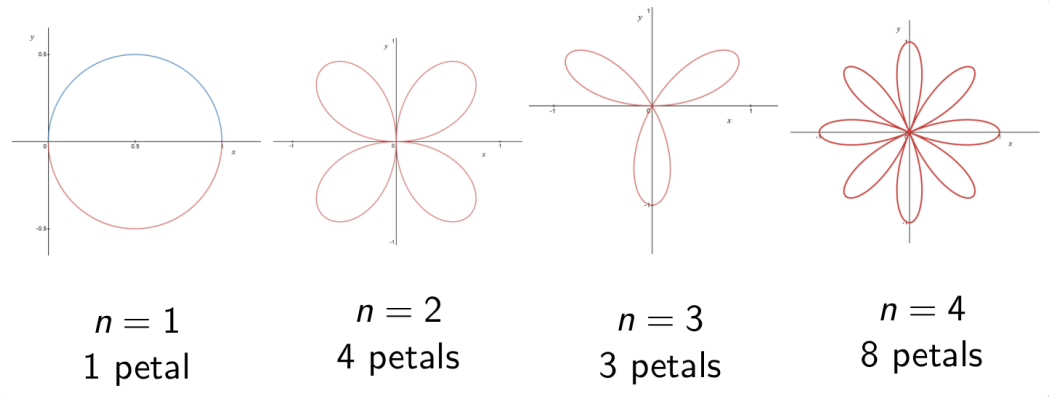
The *cusp* occurs when $c = \pm 1$

- Smaller c gives *dimple* (assuming $|c| > 0.5$)
- Larger c gives *inner loop*

02 Theory - Polar roses

Roses are polar graphs of this form:

$$r = \cos(\theta), \qquad r = \sin(2\theta), \qquad r = \sin(3\theta), \qquad r = \cos(4\theta)$$



The pattern of petals:

- $n = 2k$ (even): obtain $2n$ petals
  - These petals traversed *once*
- $n = 2k + 1$ (odd): obtain n petals
  - These petals traversed *twice*
- Either way: total-petal-traversals: always $2n$

03 Illustration

Example - Finding vertical tangents to a limaçon

Let us find the vertical tangents to the limaçon (the cardioid) given by $r = 1 + \sin\theta$.

Solution

(1) Convert to Cartesian parametric using $x = r\cos\theta$ and $y = r\sin\theta$:

$$r(\theta) = 1 + \sin\theta \quad \gg\gg \quad (x, y) = \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)$$

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(2) Compute x' and y':

$$(x'(\theta),\, y'(\theta))$$

$$\gg\gg \quad \Big(\cos\theta\cos\theta + (1+\sin\theta)(-\sin\theta),\ \cos\theta\sin\theta + (1+\sin\theta)\cos\theta\Big)$$

$$\gg\gg \quad \Big(\cos^2\theta - \sin^2\theta - \sin\theta,\ \cos\theta\,(1+2\sin\theta)\Big)$$

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(3) The vertical tangents occur when $x'(\theta) = 0$. We must double check that $y'(\theta) \neq 0$ at these points.

$$x'(\theta) = 0 \quad \gg\gg \quad \cos^2\theta - \sin^2\theta - \sin\theta = 0$$

$$\gg\gg \quad (1 - \sin^2\theta) - \sin^2\theta - \sin\theta = 0$$

Substitute $A = \sin\theta$ and observe quadratic:

$$\gg\gg \quad 1 - 2A^2 - A = 0 \quad \gg\gg \quad 2A^2 + A - 1 = 0$$

Solve:

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \gg\gg$$

$$\frac{-1 \pm \sqrt{1 - 4\cdot 2\cdot(-1)}}{2\cdot 2} \quad \gg\gg \quad \frac{1}{2},\, -1$$

Then find θ:

$$A = \sin\theta \quad \gg\gg \quad \sin\theta = \frac{1}{2},\, -1$$

$$\gg\gg \quad \theta = \frac{\pi}{6},\, \frac{5\pi}{6}\ (\text{for } 1/2) \quad \text{and} \quad \theta = \frac{3\pi}{2}\ (\text{for } -1)$$

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(4) Compute the points. In polar coordinates:

$$(r,\theta) = (1+\sin\theta,\ \theta)\Big|_{\theta=\frac{\pi}{6},\,\frac{5\pi}{6},\,\frac{3\pi}{2}}$$

$$\gg\gg \quad \left(\frac{3}{2},\frac{\pi}{6}\right),\ \left(\frac{3}{2},\frac{5\pi}{6}\right),\ \left(0,\frac{3\pi}{2}\right)$$

In Cartesian coordinates:

At $\theta = \frac{\pi}{6}$:

$$(x,y)\Big|_{\theta=\frac{\pi}{6}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{\pi}{6}}$$

$$\gg\gg \quad \left(\left(1+\frac{1}{2}\right)\frac{\sqrt{3}}{2},\ \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(\frac{3\sqrt{3}}{4},\ \frac{3}{4}\right)$$

At $\theta = \frac{5\pi}{6}$:

$$(x,y)\Big|_{\theta=\frac{5\pi}{6}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{5\pi}{6}}$$

$$\gg\gg \quad \left(\left(1+\frac{1}{2}\right)\frac{-\sqrt{3}}{2},\ \left(1+\frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(-\frac{3\sqrt{3}}{4},\ \frac{3}{4}\right)$$

At $\theta = \frac{3\pi}{2}$:

$$(x, y)\Big|_{\theta=\frac{3\pi}{2}} \quad \gg\gg \quad \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{3\pi}{2}}$$

$$\gg\gg \quad \Big((1-1)\cdot 0,\ (1-1)\cdot(-1)\Big) \quad \gg\gg \quad (0,0)$$

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(5) Correction: $(0,0)$ is a cusp!

The point $(0,0)$ at $\theta = \frac{3\pi}{2}$ is on the cardioid, but the curve is not smooth there, this is a cusp.

Still, the left- and right-sided tangents exists and are equal, so in a certain sense we could say the curve has vertical tangent at $\theta = \frac{3\pi}{2}$.

Calculus with polar curves

04 Theory - Polar tangent lines, arclength

Polar arclength formula

The arclength of the polar graph of $r(\theta)$, for $\theta \in [\alpha,\ \beta]$:

$$L \ = \ \int_{\alpha}^{\beta} \sqrt{r'(\theta)^2 + r(\theta)^2}\, d\theta$$

To derive this formula, *convert to Cartesian* with parameter θ:

$$r = r(\theta) \quad \gg\gg \quad (x, y) = (r\cos\theta,\ r\sin\theta)$$

From here you can apply the familiar arclength formula with θ in the place of t.

Extra - Derivation of polar arclength formula

Let $r = r(\theta)$ and convert to parametric Cartesian, so:

$$\begin{aligned} x(\theta) &= r(\theta)\cos\theta \\ y(\theta) &= r(\theta)\sin\theta \end{aligned}$$

Then:

$$ds \ = \ \sqrt{(x')^2 + (y')^2}\, d\theta$$

$$\begin{aligned} x' = (r\cos\theta)' \quad &\gg\gg \quad r'\cos\theta - r\sin\theta \\ y' = (r\sin\theta)' \quad &\gg\gg \quad r'\sin\theta + r\cos\theta \end{aligned}$$

Therefore:

$$\begin{aligned} (x')^2 + (y')^2 \quad \gg\gg \quad & r'^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta \\ & + r'^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta \\ \\ & = r'^2 + r^2 \end{aligned}$$

Therefore:

$$ds = \sqrt{(x')^2 + (y')^2}\, d\theta \quad \gg\gg \quad \sqrt{r'^2 + r^2}\, d\theta$$

05 Illustration

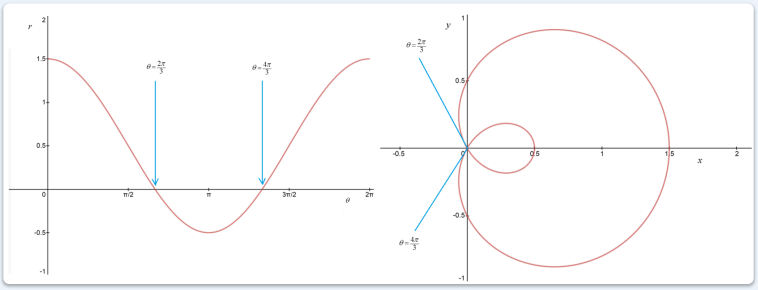
Example - Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos\theta$.

How long is the inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

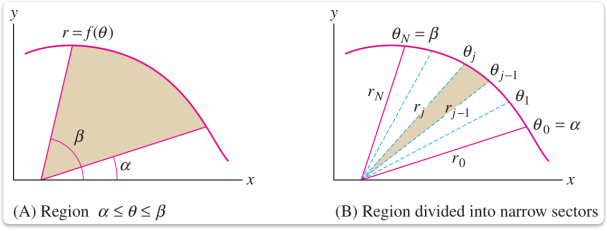
$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(-\sin\theta)^2 + \left(\frac{1}{2} + \cos\theta\right)^2}\, d\theta \quad \gg\gg \quad \int_{2\pi/3}^{4\pi/3} \sqrt{5/4 + \cos\theta}\, d\theta$$

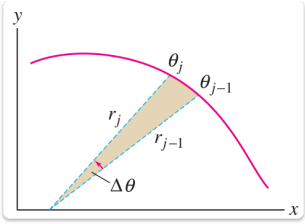
06 Theory - Polar area

Sectorial area from polar curve

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2\, d\theta$$

The "area under the curve" concept for graphs of functions in Cartesian coordinates translates to a "sectorial area" concept for polar graphs. To compute this area using an integral, we divide the region into Riemann sums of small sector slices.

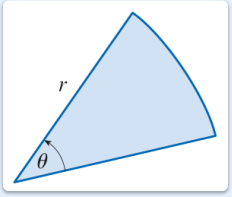




To obtain a formula for the whole area, we need a formula for the area of each sector slice.

Area of sector slice

Let us verify that the area of a sector slice is $\frac{1}{2}r^2\theta$.



Take the angle θ in radians and divide by 2π to get the *fraction of the whole disk*.

Then multiply this fraction by πr^2 (whole disk area) to get the *area of the sector slice*.

$$\left(\frac{\theta}{2\pi}\right)(\pi r^2) \quad \gg\gg \quad \frac{1}{2}r^2\theta$$

Now use $d\theta$ and $r(\theta)$ for an infinitesimal sector slice, and integrate these to get the total area formula:

$$A = \int_\alpha^\beta \frac{1}{2}r(\theta)^2\,d\theta$$

One easily verifies this formula for a circle.

Let $r(\theta) = R$ be a constant. Then:

$$\text{Area of circle} \quad = \quad \int_0^{2\pi} \frac{1}{2}R^2\,d\theta \quad \gg\gg \quad \frac{1}{2}R^2\theta\Big|_0^{2\pi} \quad \gg\gg \quad R^2\pi$$

The sectorial area *between curves*:

Sectorial area between polar curves

$$A = \int_\alpha^\beta \frac{1}{2}\left(r_1(\theta)^2 - r_0(\theta)^2\right)d\theta$$

⚠ Subtract *after* squaring, not before!

This aspect is *not* similar to the Cartesian version: $\int f - g\,dx$

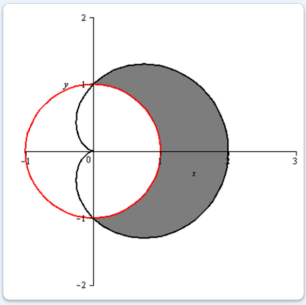
07 Illustration

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(\theta) = 1$ and the limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution

First draw the region:



The two curves intersect at $\theta = \pm\frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \le \theta \le +\frac{\pi}{2}$:

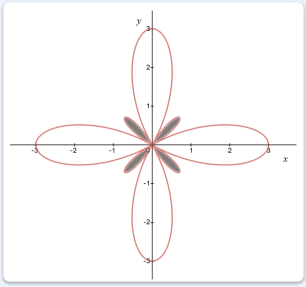
$$A = \int_a^b \frac{1}{2}(r_1^2 - r_0^2)\,d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - 1^2\Big)\,d\theta$$

$$\gg\gg \quad \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta\,d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{4}\big(1+\cos(2\theta)\big)\,d\theta$$

$$\gg\gg \quad \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \quad \gg\gg \quad 2 + \frac{\pi}{4}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

Find bounds for one small loop. Lower left loop occurs first. This loop is when $1 + 2\cos(4\theta) \le 0$.

$$1 + 2\cos(4\theta) \le 0 \qquad \gg\gg \qquad \cos(4\theta) \le -\frac{1}{2}$$

$$\gg\gg \qquad \frac{2\pi}{3} \le 4\theta \le \frac{4\pi}{3} \qquad \gg\gg \qquad \frac{\pi}{6} \le \theta \le \frac{\pi}{3}$$

Now set up area integral:

$$A = 4\int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta \quad \gg\gg \quad 4\int_{\pi/6}^{\pi/3} \frac{1}{2}\big(1 + 2\cos(4\theta)\big)^2 \, d\theta$$

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4\cos^2(4\theta) \, d\theta$$

Power-to-frequency conversion: $\cos^2 A \rightsquigarrow \frac{1}{2}(1 + \cos(2A))$ with $A = 4\theta$:

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1 + 4\cos(4\theta) + 4 \cdot \frac{1}{2}\big(1 + \cos(8\theta)\big) \, d\theta$$

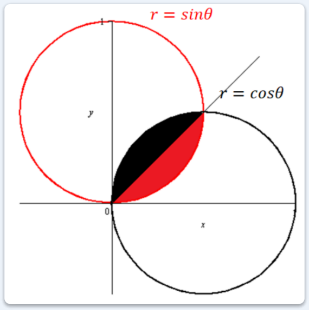
$$\gg\gg \quad 6\theta + 2\sin(4\theta) + \frac{1}{4}\sin(8\theta)\Big|_{\pi/6}^{\pi/3} \quad \gg\gg \quad \boxed{\pi - \frac{3\sqrt{3}}{2}}$$

Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta) = \sin\theta$ and the other is given by $r(\theta) = \cos\theta$.

Solution

Drawing of the overlap:



Notice: total overlap area = 2× area of red region. Bounds for red region: $0 \le \theta \le \frac{\pi}{4}$.

Area formula applied to $r(\theta) = \sin\theta$:

$$A \;=\; 2\int_{\alpha}^{\beta} \frac{1}{2} r(\theta)^2 \, d\theta \quad \gg\gg \quad 2\int_{0}^{\pi/4} \frac{1}{2}\sin^2\theta \, d\theta$$

Power-to-frequency: $\sin^2\theta \rightsquigarrow \frac{1}{2}\big(1 - \cos(2\theta)\big)$:

$$\gg\gg \quad 2\int_{0}^{\pi/4} \frac{1}{4}\big(1 - \cos(2\theta)\big) \, d\theta$$

$$\gg\gg \quad \frac{2}{4}\theta - \frac{2}{8}\sin(2\theta)\Big|_{0}^{\pi/4} \quad \gg\gg \quad \boxed{\frac{\pi}{8} - \frac{1}{4}}$$