↱ = "Division into Cases"

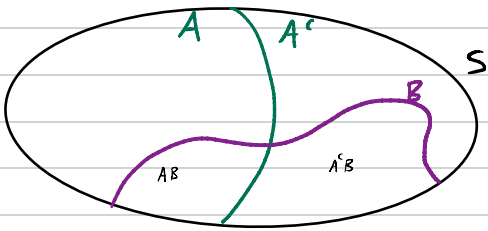
1/21

"AB" = "A∩B" = "A ∧ B"

Law of Total Probability

$$P[B] = P[A]P[B|A] + P[A^c]P[B|A^c]$$

Recall
$P[AB] = P[A]P[B|A]$
$P[A^cB] = P[A^c]P[B|A^c]$



Example: Two bins, B1 = 5 red, 4 green marbles
B2 = 4 red, 5 green marbles

Experiment: Take one from B1 put in B2. Shake B2.
Take one from B2.

Q: Find P[draw red] = ?

A:

T_R = transfer red $\quad D_R$ = draw red
T_G = transfer green $\quad D_G$ = draw green

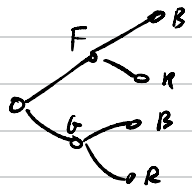
$$\text{Answer} = P[D_R] = P[T_R]P[D_R|T_R] + P[T_G]P[D_R|T_G]$$

$$= \frac{5}{9}\cdot\frac{5}{10} + \frac{4}{9}\cdot\frac{4}{10} = \boxed{\frac{41}{90}}$$

Bayes' Theorem

$$P[B|A] = P[A|B] \cdot \frac{P[B]}{P[A]} = P[B] \frac{P[A|B]}{P[A]}$$

- multistage experiments
- inferences to the middle



Example: Bins like above B1: 5r 4g, B2: 4r 5g
Friend select bin at random, ten draw from bin.
Red is drawn. Q: Prob. that he picked bin 1.

A: R, G, B_1, B_2

know: $P[B_1] = P[B_2] = \frac{1}{2}$

know: $P[R|B_1] = 5/9$ want: $P[B_1|R]$
$P[G|B_1] = 4/9$
$P[R|B_2] = 4/9$
$P[G|B_2] = 5/9$

Bayes': $P[B_1|R] = P[R|B_1] \cdot \frac{P[B_1]}{P[R]}$

$= \frac{5}{9} \cdot \frac{1/2}{?}$ need $P[R]$

$P[R] = P[B_1]P[R|B_1] + P[B_2]P[R|B_2]$
$= \frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2}\frac{4}{9} = \frac{1}{2}$

So $P[B_1|R] = \frac{5}{9} \cdot \frac{1/2}{1/2} = \boxed{\frac{5}{9}} > \frac{1}{2} = \frac{4.5}{9}$

Example:

0.5% of pop. has Covid... We have a test.
- True positive on 96% of those who have.
- False pos. on 2% of those who don't.

Bob tests positive. What are odds Bob has Covid?

A: A_N, A_P "actual" $A_N = A_A^c$
T_N, T_P "test" $T_N = T_P^c$

know: $P[T_P \mid A_P] = .96$, $P[T_P \mid A_N] = .02$

$P[A_P] = .005$ $P[A_N] = .995$

seek: $P[A_P \mid T_P]$

Bayes': $P[A_P \mid T_P] = P[T_P \mid A_P] \cdot \frac{P[A_P]}{P[T_P]}$

$= .96 \frac{.005}{P[T_P]}$ ← need

$P[T_P] = P[A_P]P[T_P \mid A_P] + P[A_N]P[T_P \mid A_N]$
$= (.005)(.96) + (.995)(.02) \approx .025 = 2.5\%$

So Ans $= (.96)\frac{.005}{(.005)(.96)+(.995)(.02)} \approx$ 19%



Independence

"product rule" when independent

$$P[B|A] = P[B] \iff P[AB] = P[A]P[B]$$

$(P[AB] = P[A]P[B|A]$ always any$)$

Example Bin: 4r 7g. Draw two marbles
(a) with replacement (b) without replacement
R_1: "red drawn first"
G_2: "green drawn second" Independent??

A: (a) $P[R_1] = 4/11$ $P[G_2] = 7/11$

$$P[R_1 G_2] = \frac{|R_1 G_2|}{|S|}, \quad |R_1 \cap G_2| = 4 \cdot 7, \quad |S| = 11 \cdot 11$$

$$\rightsquigarrow P[R_1 G_2] = \frac{4 \cdot 7}{11 \cdot 11} \overset{\checkmark}{=} \frac{4}{11} \cdot \frac{7}{11}$$ Indep.

(b) $P[R_1] = 4/11$ Seek: $P[G_2]$, $P[R_1 G_2]$

$$P[G_2] = P[R_1]P[G_2|R_1] + P[R_1^c]P[G_2|R_1^c]$$

$$= \frac{4}{11} \cdot \frac{7}{10} + \frac{7}{11} \cdot \frac{6}{10} = \frac{70}{110}$$

$$P[R_1 G_2] = P[R_1]P[G_2|R_1] = \frac{4}{11} \cdot \frac{7}{10} = \frac{28}{110}$$

$$P[R_1 G_2] \overset{?}{=} P[R_1]P[G_2] = \frac{28}{121}$$

$$\frac{28}{110} \overset{?}{=} \frac{4}{11} \cdot \frac{70}{110} \qquad 110 \neq 121$$

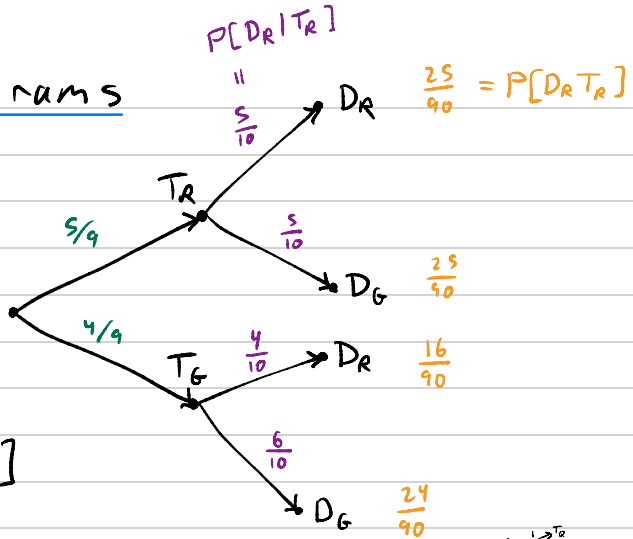
not independent!

Tree Diagrams

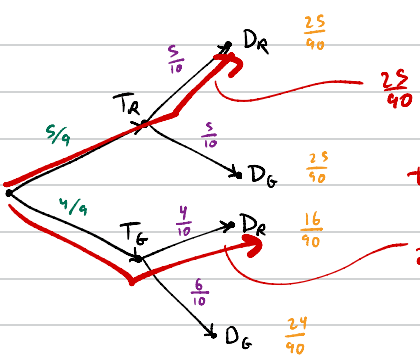
B1: 5r 4g
B2: 4r 5g

B1 → B2 → out

(a) P[draw red]
(b) P[transfer red | drew red]

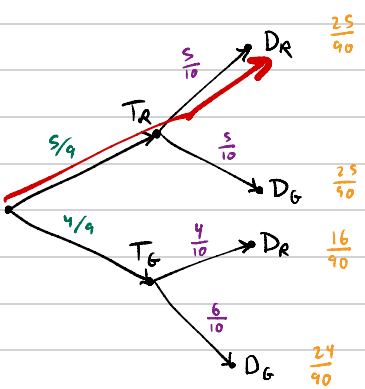


(a) P[DR]



+ → 41/90 = 5/9 · 5/10 + 4/9 · 4/10

(b)



25/90

$$P[T_R \mid D_R] = \frac{P[T_R\, D_R]}{P[D_R]}$$

$$= \frac{25/90}{41/90}$$ ← from (a)

$$= \boxed{\frac{25}{41}}$$