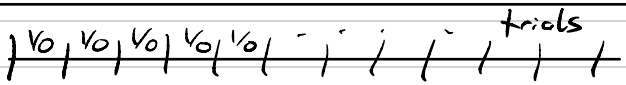
Common Discrete Families

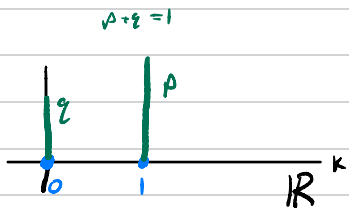
Bernoulli Process: Bernoulli, Binomial, Geometric, Pascal
Poisson Process: Poisson
↳ Exponential, Erlang are continuous

Bernoulli Process: 

E.g. flip biased coin repeatedly
HTHHTHTTTH .....
"PMF"

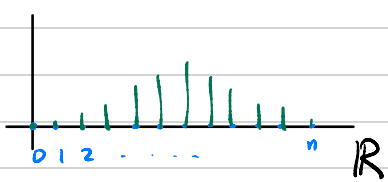
$X_i \sim Ber(p)$, indicates 1/0 of i^{th} trial

$$P_{X_i}(k) = \begin{cases} p & k=1 \\ q & k=0 \\ 0 & \text{else} \end{cases}$$



$S \sim Bin(n,p)$, counts # 1 in n trials

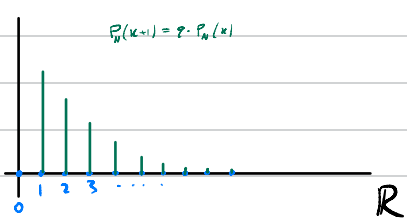
$$P_S(k) = \binom{n}{k} p^k q^{n-k} \qquad k = 0, 1, \ldots, n$$



Have relationship: $S = X_1 + \ldots + X_n$

$N \sim Geom(p)$, counts discrete wait until first 1

$$P_N(k) = q^{k-1} p \qquad k = 1, 2, 3, \ldots$$



Check: $\sum_{k=1}^{\infty} P_N(k) = \sum_{k=1}^{\infty} q^{k-1} p$

$= "\frac{a}{1-r}" = \frac{p}{1-q} = \frac{p}{p} = 1$ ✓

$\ell = 3,\ k = 5$ $\quad 01101\cdots$ $\quad 11001\cdots$ $\quad \vdots$ $\quad \binom{4}{2} p^1 \cdot p^2 \cdot q^{5-(1+2)}$

$L \sim Pasc(\ell, p)$, $\quad P_L(k) = \binom{k-1}{\ell-1} q^{k-\ell} p^{\ell} \quad k = \ell, \ell+1, \ell+2, \ldots$

counts discrete

wait until ℓ^{th} 1

Example: $L \sim Pasc(3, 1/4)$

then $P_L(8) =$ prob. 3^{rd} win on 8^{th} try

Note: In some texts: $Pasc(\ell, p) = Negbin(\ell, p)$

Example: Cubs v. Yankees in World Series

i.e. win 4 of 7 games.

Assume $p = .45$ so $q = .55$. What $P[\text{Cubs win}] = ?$

Solution: Method (a): Use Binomial:

$X \sim Bin(7, p)$ $\quad X = \#$ Cubs' wins

So $ANS = P_X(4) + P_X(5) + P_X(6) + P_X(7)$

Have: $P_X(k) = \binom{7}{k} p^k q^{7-k}$

$\rightsquigarrow \binom{7}{4} p^4 q^3 + \binom{7}{5} p^5 q^2 + \binom{7}{6} p^6 q^1 + \binom{7}{7} p^7 q^0$

$\rightsquigarrow \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} p^4 q^3 + \frac{7 \cdot 6}{2} p^5 q^2 + \frac{7}{1} p^6 q^1 + 1 \cdot p^7 q^0$

$\rightsquigarrow p^4 (35 q^3 + 21 p^1 q^2 + 7 p^2 q + p^3) \rightsquigarrow 0.39$

Method (b):

$Y \sim \text{Pasc}(4, p)$ $\quad Y = \#$ games till Cubs' 4^{th} w.n.

$\text{ANS} = P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$

$P_Y(k) = \binom{k-1}{4-1} p^4 q^{k-4}$

$\rightsquigarrow \binom{3}{3} p^4 q^0 + \binom{4}{3} p^4 q^1 + \binom{5}{3} p^4 q^2 + \binom{6}{3} p^4 q^3$

$\rightsquigarrow 1 \cdot p^4 q^0 + \frac{4}{1} p^4 q^1 + \frac{5 \cdot 4}{2} p^4 q^2 + \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} p^4 q^3$

$\rightsquigarrow p^4 (1 + 4q + 10q^2 + 20q^3) \rightsquigarrow 0.39$

Expectation

$E[X]$ = avg. of X, weights: $P_X(k)$ $= \mu$

$$= \sum_k k\, P_X(k) \qquad = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

aka "expected value"

Say $H \overset{X}{\rightsquigarrow} 17$, $T \rightsquigarrow -3$

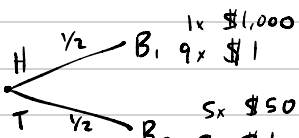
$E[X] = 17 \cdot \frac{1}{2} + (-3) \cdot \frac{1}{2}$

20 — 10 strongly agree, 7 agree

10 — 0 neutral, -7 disagree

0 — -10 strongly disagree

Example: 1. Flip fair coin
2. Draw from Bin



Q: · Expected winning?
· If costs $50 to play, should you?

Solution: Define X = winnings.

Find $P_X(k)$:

X	1	50	1,000
P_X	$7/10$	$1/4$	$1/20$

$P[X=1{,}000 \mid B_1] = .1$ $\quad P[X=50 \mid B_2] = .5$
$P[X=1 \mid B_1] = .9$ $\quad P[X=1 \mid B_2] = .5$

$$P_X(1) = P[X=1] = \tfrac{1}{2}\cdot\tfrac{9}{10} + \tfrac{1}{2}\cdot\tfrac{5}{10} = \tfrac{7}{10}$$
$$P_X(50) = \tfrac{1}{2}\cdot\tfrac{5}{10}$$
$$P_X(1{,}000) = \tfrac{1}{2}\cdot\tfrac{1}{10}$$

$$E[X] = \sum k\, P_X(k)$$
$$= 1\cdot\tfrac{7}{10} + 50\cdot\tfrac{1}{4} + 1{,}000\cdot\tfrac{1}{20}$$
$$= \boxed{63.2}$$

See that $63.2 > 50$.
With ∞ games you come ahead.
With 1 game... might go bankrupt

Example:

x	1	2	3	4	5
P_x	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{2}{7}$

Find $E[|X-2|]$.

Solution: Method 1: $Y = |X-2|$

$$P_Y(k) = \begin{cases} 1/14 & k=0, \ x=2 \\ \frac{1}{7}+\frac{3}{14} & k=1, \ x=1,3 \\ 2/7 & k=2, \ x=4 \\ 2/7 & k=3, \ x=5 \\ 0 & k=\text{else} \end{cases}$$

$$\rightsquigarrow E[Y] = 0\cdot\frac{1}{14} + 1\cdot\frac{5}{14} \cdot 2\cdot\frac{2}{7} + 3\cdot\frac{2}{7} = \boxed{\frac{25}{14}}$$

Method 2:

x	1	2	3	4	5
P_x	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{2}{7}$
Y	1	0	1	2	3

$$E[Y] = 1\cdot\frac{1}{7} + 0\cdot\frac{1}{14} + 1\cdot\frac{3}{14} + 2\cdot\frac{2}{7} + 3\cdot\frac{2}{7} = \boxed{\frac{25}{14}}$$

Variance

$$\text{Var}[X] = E[(X-\mu)^2] = \sum_k (k-\mu)^2 P_x(k) = \int_{-\infty}^{\infty} (x-\mu)^2 f_x(x)\,dx$$

$$= E[X^2] - E[X]^2$$

= "short formula"

σ_X = standard deviation = $\sqrt{\text{Var}[X]}$

Example:

X	1	2	3
p_X	$1/7$	$2/7$	$4/7$

$Y = \frac{1}{1+X}$ Find $Var[Y] = Var\left[\frac{1}{1+X}\right]$.

Solution: Use $Var[Y] = E[Y^2] - E[Y]^2$

X	1	2	3
p_X	$1/7$	$2/7$	$4/7$
Y	$1/2$	$1/3$	$1/4$
Y^2	$1/4$	$1/9$	$1/16$

$$E[Y] = \frac{1}{2}\cdot\frac{1}{7} + \frac{1}{3}\cdot\frac{2}{7} + \frac{1}{4}\cdot\frac{4}{7} = \frac{13}{42}$$

$$E[Y^2] = \frac{1}{4}\cdot\frac{1}{7} + \frac{1}{9}\cdot\frac{2}{7} + \frac{1}{16}\cdot\frac{4}{7} = \frac{13}{126}$$

$$\rightsquigarrow Var[Y] = \frac{13}{126} - \left(\frac{13}{42}\right)^2 = \boxed{7.4\times10^{-3}}$$