W11 Notes

Review: True/False

TRUE or FALSE:

(a) Suppose $\text{Cov}[X, Y] = 0.05$. It is possible that $\rho_{X,Y} = 0.05$.

(b) Suppose $\text{Cov}[X, Y] = 0.05$. It is possible that X and Y have a strong linear relationship.

(c) Suppose $\text{Cov}[X, Y] = 0$. It is possible that X and Y are *not* independent.

(d) Suppose X and Y are *not* independent. It is possible that $\text{Cov}[X, Y]$ is equal to 0.

(e) Suppose X and Y *are* independent. $\text{Cov}[X, Y]$ must be equal to 0.

Review: Conditional probability

Recall some items related to conditional probability.

Conditioning definition:

$$P[- \mid A] \;=\; \frac{P[- \cap A]}{P[A]}$$

Multiplication rule:

$$P[AB] \;=\; P[B \mid A]\, P[A]$$

Division into Cases / Total Probability:

$$P[B] \;=\; P[B \mid A_1]\, P[A_1] + \cdots + P[B \mid A_n]\, P[A_n]$$

Conditional distribution

01 Theory

Conditional distribution - fixed event

Suppose X is a random variable, and suppose $A \subset \mathbb{R}$. The distribution of X **conditioned on** A describes the probabilities of values of X given knowledge that $X \in A$.

Discrete case:

$$P_{X|A}(k) \;=\; \begin{cases} \dfrac{1}{P[A]} P_X(k) & k \in A \\ 0 & k \notin A \end{cases}$$

Continuous case:

$$f_{X|A}(x) \;=\; \begin{cases} \dfrac{1}{P[A]} f_X(x) & x \in A \\ 0 & x \notin A \end{cases}$$

There is also a conditional CDF, of which this conditional PDF is the derivative:

$$F_{X|A}(x) \;=\; P[X \le x \mid A], \qquad f_{X|A}(x) \;=\; \frac{d}{dx} F_{X|A}(x)$$

The Law of Total Probability has versions for distributions:

$$\begin{aligned} P_X(k) &= P_{X|A_1}(k)\, P[A_1] + \cdots + P_{X|A_n}(k)\, P[A_n] \\ f_X(x) &= f_{X|A_1}(x)\, P[A_1] + \cdots + f_{X|A_n}(x)\, P[A_n] \end{aligned}$$

Conditional distribution - variable event

Suppose X and Y are any two random variables. The **distribution of X conditioned on Y** describes the probabilities of values of X in terms of y, given knowledge that $Y = y$.

Discrete case:

$$\begin{aligned} P_{X|Y}(k|\ell) &= P[X = k \mid Y = \ell] \\ &= \frac{P_{X,Y}(k,\ell)}{P_Y(\ell)} \qquad \text{(assuming } P_Y(\ell) \neq 0\text{)} \end{aligned}$$

Continuous case:

$$f_{X|Y}(x|y) \;=\; \frac{f_{X,Y}(x,y)}{f_Y(y)} \qquad \text{(assuming } f_Y(y) \neq 0\text{)}$$

Remember: $P_{X,Y}(k,\ell)$ is the probability that "$X = k$ and $Y = \ell$."

Sometimes it is useful to have the formulas rewritten like this:

$$\begin{aligned} P_{X,Y}(k,\ell) &= P_{X|Y}(k|\ell)\, P_Y(\ell) \\ f_{X,Y}(x,y) &= f_{X|Y}(x|y)\, f_Y(y) \end{aligned}$$

Extra - Deriving $f_{X|Y}(x|y)$

The density $f_{X|Y}$ ought to be such that $f_{X|Y}(x|y)\,dx$ gives the probability of $X \in [x, x+dx]$, given knowledge that $Y \in [y, y+dy]$. Calculate this probability:

$$P\Big[x \le X \le x+dx \;\Big|\; y \le Y \le y+dy\Big]$$

$$\gg\gg \quad \frac{P\Big[x \le X \le x+dx,\; y \le Y \le y+dy\Big]}{P\Big[y \le Y \le y+dy\Big]}$$

$$\gg\gg \quad \frac{f_{X,Y}(x,y)\,dx\,dy}{f_Y(y)\,dy}$$

$$\gg\gg \quad \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

02 Illustration

Example - Conditional PMF, variable event, via joint density

Suppose X and Y have joint PMF given by:

$$P_{X,Y}(k,\ell) = \begin{cases} \dfrac{k+\ell}{21} & k=1,2,3;\ \ell=1,2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P_{X|Y}(k|\ell)$ and $P_{Y|X}(\ell,k)$.

Solution

Marginal PMFs:

$$P_X(k) = \frac{2k+3}{21}, \qquad k=1,2,3$$

$$P_Y(\ell) = \frac{\ell+2}{7}, \qquad \ell=1,2$$

Assuming $\ell = 1$ or 2, for each $k = 1$, 2, 3 we have:

$$P_{X|Y}(k|\ell) = \frac{P_{X,Y}(k,\ell)}{P_Y(\ell)} \quad \gg\gg \quad \frac{k+\ell}{3\ell+6}$$

Assuming $k = 1$, 2, or 3, for each $\ell = 1, 2$ we have:

$$P_{Y|X}(\ell|k) = \frac{P_{Y,X}(\ell,k)}{P_X(k)} \quad \gg\gg \quad \frac{k+\ell}{2k+3}$$

Conditional expectation

03 Theory

Expectation conditioned by a fixed event

Suppose X is a random variable and $A \subset \mathbb{R}$. The **expectation of X conditioned on A** describes the typical value of X given the hypothesis that $X \in A$ is known.

Discrete case:

$$E[\,X \mid A\,] = \sum_k k\, P_{X|A}(k)$$

$$E[\,g(X) \mid A\,] = \sum_k g(k)\, P_{X|A}(k)$$

Continuous case:

$$E[\,X \mid A\,] = \int_{-\infty}^{+\infty} x\, f_{X|A}(x)\, dx$$

$$E[\,g(X) \mid A\,] = \int_{-\infty}^{+\infty} g(x)\, f_{X|A}(x)\, dx$$

Conditional variance:

$$\mathrm{Var}[\,X \mid A\,] \;=\; E\Big[(X-\mu_{X|A})^2 \mid A\Big] \quad = \quad E\big[X^2 \mid A\big] - \mu_{X|A}^2$$

Division into Cases / Total Probability applied to expectation:

$$E[X] \;=\; E[\,X \mid A_1\,]\,P[A_1] + \cdots + E[\,X \mid A_n\,]\,P[A_n]$$

Linearity of conditional expectation:

$$E[\,aX_1 + bX_2 + c \mid Y = y\,] = a\,E[\,X_1 \mid Y = y\,] + b\,E[\,X_2 \mid Y = y\,] + c$$

Extra - Proof: Division of Expectation into Cases

We prove the discrete case only.

1. Expectation formula:

$$E[X] \;=\; \sum_k k\,P_X(k)$$

2. Division into Cases for the PMF:

$$P_X(k) \;=\; \sum_{i=1}^{n} P_{X|A_i}(k)\,P[A_i]$$

3. Substitute in the formula for $E[X]$:

$$\sum_k k\,P_X(k) \quad \gg\gg \quad \sum_k k \sum_{i=1}^{n} P_{X|A_i}(k)\,P[A_i]$$

$$\gg\gg \quad \sum_{i=1}^{n} P[A_i] \sum_k k\,P_{X|A_i}(k)$$

$$\gg\gg \quad \sum_{i=1}^{n} P[A_i]\,E[\,X \mid A_i\,]$$

Expectation conditioned by a variable event

Suppose X and Y are any two random variables. The **expectation of X conditioned on $Y = y$** describes the typical of value of X in terms of y, given the hypothesis that $Y = y$ is known.

Discrete case:

$$E[\,X \mid Y = y\,] \;=\; \sum_k k\,P_{X|Y}(k|y) \quad (k \text{ over all poss. vals.})$$

$$E[\,g(X,Y) \mid Y = y\,] \;=\; \sum_k g(k,y)\,P_{X|Y}(k|y)$$

Continuous case:

$$E[\,X \mid Y=y\,] \;=\; \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx$$

$$E[\,g(X,Y) \mid Y=y\,] \;=\; \int_{-\infty}^{+\infty} g(x,y)\, f_{X|Y}(x|y)\, dx$$

05 Illustration

Example - Conditional PMF, fixed event, expectation

Suppose X measures the lengths of some items and has the following PMF:

$$P_X(k) \quad = \quad \begin{cases} 0.15 & k = 1,2,3,4 \\ 0.1 & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Let $L =$ "$X \geq 5$", an event.

(a) Find the conditional PMF of X given that L is known.

(b) Find the conditional expected value and variance of X given L.

Solution

(a)

Conditional PMF formula with $k \in L$ plugged in:

$$P_{X|L}(k) \quad = \quad \begin{cases} \dfrac{P_X(k)}{P[L]} & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

Compute $P[L]$ by adding cases:

$$P[L] \;=\; \sum_{k=5}^{8} P_X(k) \quad \ggg \quad 0.4$$

Divide nonzero PMF entries by 0.1:

$$P_{X|L}(k) \;=\; \begin{cases} 0.25 & k = 5,6,7,8 \\ 0 & \text{otherwise} \end{cases}$$

(b)

Find $E[\,X \mid L\,]$:

$$E[\,X \mid L\,] \;=\; \sum_{k=5}^{8} k\, P_{X|L}(k)$$

$$\ggg \quad 5 \cdot (0.25) + 6 \cdot (0.25) + 7 \cdot (0.25) + 8 \cdot (0.25)$$

$$\ggg \quad 6.5\,\text{min}$$

Find $E[\,X^2 \mid L\,]$:

$$E[\,X^2 \mid L\,] \;=\; \sum_{k=5}^{8} k^2\, P_{X|L}(k)$$

$$\gg\gg \quad 5^2 \cdot (0.25) + 6^2 \cdot (0.25) + 7^2 \cdot (0.25) + 8^2 \cdot (0.25)$$

$$\gg\gg \quad 43.5\,\text{min}^2$$

Find $\text{Var}[\,X \mid L\,]$ using "short form" with conditioning:

$$\text{Var}[\,X \mid L\,] \quad = \quad E[\,X^2 \mid L\,] - E[\,X \mid L\,]^2 \quad \gg\gg \quad 1.25\,\text{min}^2$$

04 Illustration

Example - Conditional expectations from joint density

Suppose X and Y are random variables with joint density given by:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{y} e^{-x/y} e^{-y} & x, y \in (0,\infty) \\ 0 & \text{otherwise} \end{cases}$$

Find $E[\,X \mid Y = y\,]$.

Solution

(1) Derive the marginal density $f_Y(y)$:

$$f_Y(y) \quad \gg\gg \quad \int_0^{+\infty} \tfrac{1}{y} e^{-x/y} e^{-y}\, dx$$

$$\gg\gg \quad -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty} \quad \gg\gg \quad e^{-y}$$

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(2) Use $f_Y(y)$ to compute $f_{X|Y}(x|y)$:

$$f_{X|Y}(x|y) \quad \gg\gg \quad \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\gg\gg \quad \tfrac{1}{y} e^{-x/y} e^{-y} \cdot (e^{-y})^{-1} \quad \gg\gg \quad \tfrac{1}{y} e^{-x/y}$$

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(3) Use $f_{X|Y}(x|y)$ to calculate expectation conditioned on the variable event:

$$E[X \mid Y = y] \quad \gg\gg \quad \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx$$

$$\gg\gg \quad \int_0^{\infty} \tfrac{x}{y} e^{-x/y}\, dx \quad \gg\gg \quad y$$

05 Theory - extra (non-examinable)

Expectation conditioned by a random variable

Suppose X and Y are any two random variables. The **expectation of X conditioned on Y** is a random variable giving the typical value of X on the assumption that Y has value determined by an outcome of the experiment.

$$E[\,X \mid Y\,] \;=\; g(Y) \quad \text{where } \; g(y) = E[\,X \mid Y = y\,]$$

In other words, start by defining a function $g(y)$:

$$\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ y &\mapsto E[\,X \mid Y = y\,] \end{aligned}$$

Now $E[\,X \mid Y\,]$ is defined as the composite random variable $g(Y)$.

Considered as a random variable, $E[\,X \mid Y\,]$ takes an outcome $s \in S$, computes $Y(s)$, sets $y = Y(s)$, then returns the expectation of X conditioned on $Y = y$.

Notice that X is *not* evaluated at s, only Y is.

Because the value of $E[\,X \mid Y\,]$ depends only on $Y(s)$, and not on any additional information about s, it is common to *represent* a conditional expectation $E[\,X \mid Y\,]$ using only the function g.

Iterated Expectation

$$E[\;E[X \mid Y]\;] = E[X]$$

Proof of Iterated Expectation, discrete case

$$\begin{aligned} E[\,E[\,X \mid Y\,]\,] \;&=\; \sum_{\ell} E[X \mid Y = \ell]\, P_Y(\ell) \\ &= \sum_{\ell} \sum_{k} k\, P_{X|Y}(k|\ell)\, P_Y(\ell) \\ &= \sum_{k} k \sum_{\ell} P_{X,Y}(k,\ell) \\ &= \quad \sum_{k} k\, P_X(k) = E[X] \end{aligned}$$

06 Illustration

Example - Conditional Expectation Variable

Let N denote the number of customers that enter a store on a given day.

Let X_i denote the amount spent by the i^{th} customer.

Assume that $E[N] = 50$ and $E[X_i] = \$8$ for each i.

What is the expected total spend of all customers in a day?

Solution

A formula for the total spend is $X = \sum_{i=1}^{N} X_i$.

By Iterated Expectation, we know $E[X] = E[\,E[\,X \mid N\,]\,]$.

Now compute $E[X \mid N]$ as a function of N:

$$
\begin{aligned}
E[X \mid N = n] \quad &\gg\gg \quad E\left[\left(\sum_{i=1}^{N} X_i\right) \mid N = n\right] \\
&\gg\gg \quad E\left[\left(\sum_{i=1}^{n} X_i\right) \mid N = n\right] \\
&\gg\gg \quad \sum_{i=1}^{n} E[X_i \mid N = n] \\
&\gg\gg \quad \sum_{i=1}^{n} E[X_i] \quad \gg\gg \quad 8n
\end{aligned}
$$

Therefore $g(n) = 8n$ and $g(N) = 8N$ and $E[\,X \mid N\,] = 8N$.

Then by Iterated Expectation, $E[X] = E[8N] = 8E[N] = \$400$.