# "Statistical Inference Theory"

## Significance Testing
"unary" hypothesis testing

### Significance Test
- Null hypothesis $H_0$ (background assumption)
  - Identify a Claim $\rightsquigarrow$ $H_0$ less likely
    wish to invalidate $H_0$ in favor of Claim

- Rejection Region $R$ ("decision rule")
  - Need a "decision statistic" $X$, an RV
  - Write $R$ in terms of $X$ values
  - "Significance level" $\alpha$
  - $H_0 \rightsquigarrow R$ less likely, Claim $\rightsquigarrow R$ more likely

- $P[R \mid H_0]$
  - compute somehow
  - usually infer from $f_{X|H_0}$, $P_{X|H_0}$
  - adjust $R$ so that $P[R \mid H_0] = \alpha$.

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$\alpha = P[R \mid H_0]$
$= P[\text{reject } H_0 \mid H_0 \text{ is true}]$ = "Type I error"

| | $H_0$ is true | $H_0$ is false |
|---|---|---|
| Maintain $H_0$ | Good call | Type II error |
| Reject $H_0$ | Type I error | Good call |

"$H_0$ innocent until proven guilty"

**Example:** Your friend gives you a die, worries that it's weighted to come up 2 more often. Design a sign test on data of 20 rolls to decide whether weighted, $\alpha = 5\%$.

**Solution:**

- $H_0$ = die is fair
  - Claim: 2 is more likely/frequent than $1/6$
- $R$ = ? (need to compute)
  - $X$ = # 2's. $X|H_0 \sim Bin(20, 1/6)$
  - $R = \{x \mid x \geq r\}$ some $r$ (need $r$)
- $P[R|H_0] = P[X \geq r|H_0] = \alpha = .05$

Use $P_{X|H_0}(k) = \binom{20}{k} (1/6)^k (5/6)^{20-k}$

= prob. of $k$ twos

Now, rewrite: $P[X < r|H_0] = 0.95$

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | ... |
|---|---|---|---|---|---|---|---|---|---|
| $F_{X\|H_0}$ | .026 | .130 | .329 | .567 | .769 | .898 | .963 | .989 | |

$\Rightarrow$ choose $r = 6$

$\rightsquigarrow P[X \geq 6|H_0] < .05$

but $P[X \geq 5|H_0] \not< .05$

$$\boxed{R = \{X \mid X \geq 6\}}$$

**Example:** AC circuit nominal 130 V, $\sigma = 2.1$. Design a two-tail sign. test based on 40 measurements to determine whether the true avg. voltage is 130, $\alpha = 2\%$.

**Solution:**

- $H_0$ = "$E[V] = 130$"
  Claim $E[V] \neq 130$
- $X = M_{40}(V)$ $\quad V \sim ??$
- $R = |X - 130| \geq c$ find $c$ s.t.

$$P[|X-130| \geq c] = 0.02$$

Use Chebyshev:

$$P[|X-130| \geq c] \leq \frac{\sigma_X^2}{c^2} = \frac{\sigma_V^2}{40 \cdot c^2}$$

$$\sigma_V = 2.1$$

Solve $\frac{(2.1)^2}{40 \cdot c^2} = 0.02 \rightsquigarrow c = 2.348$

$$\Rightarrow \boxed{R = \{X \mid X < 127.65 \text{ OR } X > 132.35\}}$$

# Binary Hypothesis Testing

Ingredients of test:

- $H_0$ & $H_1 = H_0^c$
  - Maybe know $P[H_0]$, $P[H_1]$ (MAP, MC)
- $A_0$ & $A_1 = A_0^c$
  - $H_0 \rightsquigarrow A_0$ likely, $H_1 \rightsquigarrow A_1$ likely
  - Test rule: decide $H_0$ or $H_1$
  - Write $A_0$ using decision statistic $X$
- Choose a design type: MAP, ML, MC

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## MAP Design "maximum a posteriori probability"

Suppose we know $P[H_0]$, $P[H_1]$
and $P_{X|H_0}(x)$, $P_{X|H_1}(x)$ (or $f_{X|H_0}$ ...)

MAP design: $A_0$ = set of all $x$ such that

discrete: $P_{X|H_0}(x) \cdot P[H_0] \geq P_{X|H_1}(x) \cdot P[H_1]$

cts: $f_{X|H_0}(x) \cdot P[H_0] \geq f_{X|H_1}(x) \cdot P[H_1]$

$$A_1 = \{x \in \mathbb{R} \mid x \notin A_0\}$$

This minimizes the total probability of error.

# ML Design

(Don't know priors $P[H_0]$, $P[H_1]$.)

ML design: $A_0$ = all $x$ such that

discrete: $P_{X|H_0}(x) \geq P_{X|H_1}(x)$

cts: $f_{X|H_0}(x) \geq f_{X|H_1}(x)$

Like: simplify MAP by either
view as:
- set $P[H_0] = P[H_1]$ and cancel
- ignore priors

Can use as frequentist!

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Type I error, false alarm rate:

$$P_{FA} = P[A_1 \mid H_0]$$

Type II error, miss rate:

$$P_{Miss} = P[A_0 \mid H_1]$$

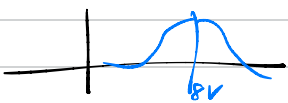
Total error rate: $P_{FA} \cdot P[H_0] + P_{Miss} \cdot P[H_1]$

**Example:** ML Test, smoke detectors

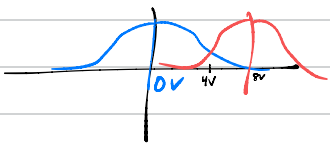
Suppose sensor makes 8V when no smoke,
0V when smoke

But... background noise: $\mathcal{N}(0, 3^2)$

Smoke:




No smoke:




- Design ML test to sound alarm.
- What are the three error rates?

**Solution:**

$$X|H_0 \sim \mathcal{N}(0, 3^2)$$

$$X|H_1 \sim \mathcal{N}(8, 3^2)$$

$$f_{X|H_0} = \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2}, \quad f_{X|H_1} = \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

ML condition:

$$\frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \gtrless \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

$$\rightarrow \quad x \leq 4 = A_0, \quad x > 4 = A_1$$

$X = 3 \cdot Z + 0$ (when $H_0$)

Type I error:

$$P_{FA} = P[A_1 \mid H_0] \rightsquigarrow P(X > 4 \mid H_0]$$

$$\rightsquigarrow 1 - P\left[\frac{X-0}{3} \leq \frac{4}{3} \mid H_0\right]$$

$$\rightsquigarrow 1 - P[Z \leq 1.33]$$

$$\rightsquigarrow \boxed{0.0912}$$

Type II error:

$$P_{Miss} = P[A_0 \mid H_1] \rightsquigarrow P[X \leq 4 \mid H_1]$$

$$\rightsquigarrow P\left[\frac{X-8}{3} \leq \frac{4-8}{3} \mid H_1\right]$$

$$\rightsquigarrow P[Z \leq -1.33]$$

$$\rightsquigarrow \boxed{0.0912}$$

$$P_{ERR} = P_{FA} \cdot 0.5 + P_{Miss} \cdot 0.5$$

$$\rightsquigarrow \boxed{0.0912}$$

**Example:** Extend above smoke detector test as a MAP design assuming chance of smoke = 5%. Find all error rates.

**Solution:** So $P[H_1] = .05$, $\rightsquigarrow P[H_0] = .95$

MAP condition: $f_{X|H_0}(x)\,P[H_0] \geq f_{X|H_1}(x)\,P[H_1]$

$$\rightsquigarrow \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2}(0.95) \geq \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}(0.05)$$

$$\rightsquigarrow -\frac{1}{2}\left(\frac{x-0}{3}\right)^2 \geq -\frac{1}{2}\left(\frac{x-8}{3}\right)^2 + \ln\left(\frac{0.05}{0.95}\right)$$

$$\rightsquigarrow x^2 \leq (x-8)^2 - 18\ln\left(\frac{.05}{.95}\right)$$

$$\rightsquigarrow \{x \leq 7.31\} = A_0 \qquad A_1 = \{x > 7.31\}$$

Sound alarm when $x > 7.31$.

$P_{FA} = P[X > 7.31 \mid H_0] = .007$ $\quad 1 - P[Z \leq 2.44]$

$P_{Miss} = P[X \leq 7.31 \mid H_1] = .41$

$P_{ERR} = P_{FA} \cdot P[H_0] + P_{Miss} \cdot P[H_1] \rightsquigarrow (.007)(.95) + (.41)(.05)$

$\rightsquigarrow 0.027$

**Example:** Extend now to minimum cost (MC) design assuming cost of Miss = 50x cost of FA.

**Solution:** $C_{10} = 1, \quad C_{01} = 50$

"10" read right-to-left, cost of $A_1 | H_0$

MC condition: $A_0 = x$ such that:

$$f_{X|H_0}(x)\, P[H_0]\, C_{10} \gtrless f_{X|H_1}(x)\, P[H_1]\, C_{01}$$

$$\rightsquigarrow \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} (.95)(1) \gtrless \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2} (.05)(50)$$

$$\rightsquigarrow \{x \leq 2.91\} = A_0$$

Type I error: $P_{FA} = P[X > 2.91 | H_0]$

$\rightsquigarrow 0.17$

Type II error: $P_{Miss} = P[X \leq 2.91 | H_1]$

$\rightsquigarrow 0.045$

$P_{ERR} = (.17)(.95) + (.045)(.05) = .16$

Total cost of error:

$$P_C(c) = \begin{cases} .00224 & c = 50 \\ .1577 & c = 1 \\ .8401 & c = 0 \end{cases} \rightsquigarrow E[C] = .27$$